Introduction and Basic Concepts

1.1 Introduction

We are embarking on a study of heat and mass transfer. Heat is defined as *energy in transit*. Heat itself cannot be seen, but its effect can be felt and measured as a property called temperature. Heat transfer occurs whenever two bodies at different temperatures are brought in contact with each other or, whenever there is a temperature gradient within a body. Science of heat transfer involves the study of principles that govern and the methods that determine the rate of heat transfer. Often, we are also interested in the spatial temperature distribution within a body causing that heat transfer.

In this chapter, we will primarily give an introduction to the three different modes of heat transfer, viz., conduction, convection and radiation and the corresponding rate equations that govern these processes. We will mention about the fundamental laws generally applied in heat transfer analysis, and analogies with other transport processes. We will also indicate a few areas where the science of heat transfer and mass transfer finds its applications.

1.2 Thermodynamics and Heat Transfer

In the course on thermodynamics you have studied the interaction of heat and work and the laws of thermodynamics. First law deals with energy balances and leads to the concept of enthalpy, whereas second law deals with availability balances and determines the direction in which heat energy will flow and leads to the concept of entropy. In fact, it is the second law which says that heat flows from a location of high temperature to a location of low temperature. Then, you may be wondering as to what is the need for a separate science of heat transfer. The answer is, thermodynamics deals with equilibrium (quasistatic) processes; total heat transferred from one equilibrium state to another equilibrium state can be easily calculated by the laws of thermodynamics. However, the rate of heat transfer and the temperature variation with time and position cannot be calculated by these laws alone and to do this, we need the laws of heat transfer. As already stated, heat transfer requires a temperature gradient, i.e. essentially we need to know the temperature distribution within a body and the laws connecting the rate of heat transfer with this temperature gradient.

In the chapters to follow, we will study these laws as well as their applications.

1.3 Applications of Heat Transfer

Heat transfer is an important branch of thermal science which has applications in diverse fields of engineering.

(a) **Mechanical engineering** In boilers, heat exchangers, turbine systems, internal combustion engines etc.

(b) **Metallurgical engineering** In furnaces, heat treatment of components etc.

(c) **Electrical engineering** Cooling systems for electric motors, generators, transformers etc.

(d) **Chemical engineering** In process equipments used in refineries, chemical plants etc.

(e) **Nuclear engineering** In removal of heat generated by nuclear fission using liquid metal coolants, design of nuclear fuel rods against possible burnout etc.

(f) **Aerospace engineering & space technology** In the design of aircraft systems and components, rockets, missiles etc.

(g) **Cryogenic engineering** In the production, storage, transportation and utilisation of cryogenic liquids (at very low temperatures ranging from 100 K to 4 K or even lower) for various industrial, research and defence applications.

(h) **Civil engineering** In the design of suspension bridges, railway tracks, airconditioning and insulation of buildings etc.

There are numerous other applications where principles and methods of heat transfer are widely applied and directly affect our lives.

1.4 Fundamental Laws of Heat Transfer

Science of heat transfer, of course, operates within the limits of the laws of thermodynamics. Additionally, subsidiary laws relating to fluid flow and rate equations for different modes of heat transfer are also required for a complete solution.

Fundamental laws governing heat transfer are enumerated below:

(i) First law of thermodynamics—gives conservation of energy,

(ii) Second law of thermodynamics—gives direction of heat flow,

(iii) Equation of continuity—gives conservation of mass,

(iv) Equation of flow—Newton's Second law of motion—Navier Stokes' equation,

(v) Rate equations governing the three modes of heat transfer,

   (a) Conduction—Fourier's law of conduction

   (b) Convection—Newton's law of cooling

   (c) Radiation—Stefan Boltzmann's law

(vi) Empirical relations for fluid properties such as specific heat, thermal conductivity, viscosity etc.,

(vii) Equation of state for the fluid.

As we go along, we will have occasion to see how these laws are applied to the problems at hand. Of course, which laws have to be applied in a given situation depends on the specific problem; aim of this course is specifically to give such an insight to the student.

1.5 Analogies with Other Transport Processes

We will take a little diversion from our main stream of thought here. It is apparent that there is so much symmetry in nature and natural processes that occur, that sometimes, by observing one process, one may be able to predict the outcome of another similar process. For example, consider the three important processes of transport of energy (heat), transport of momentum and transport of mass. We know that in nature, flow occurs spontaneously from a higher potential to a lower potential: electricity flows from a higher voltage potential to a lower voltage potential and water flows from a higher datum level (pressure) to a lower datum level. Similarly, for the aforementioned three transport processes, we can observe following:

(i) Transport of heat energy—occurs from a higher temperature level to a lower temperature level.

(ii) Transport of momentum—occurs from a higher velocity level to a lower velocity level.

(iii) Transport of mass—occurs from a higher concentration level to a lower concentration level.

In other words, we can say that the driving potential for heat transfer is the temperature gradient, driving potential for momentum transfer is the velocity gradient and the driving potential for mass transfer is the concentration gradient. Therefore, we can feel that the governing equations for these processes must have some similarity. In fact, it will be seen later, when we study conduction, convection and mass transfer, that such a similarity does exist in the governing equations for these processes. Therefore, knowing the solutions for one transport process, we will be able to predict the solutions for another transport process by analogy.

1.6 Modes of Heat Transfer

Generally, for convenience of analysis, we consider heat transfer in three different modes: conduction, convection and radiation.

1.6.1 Conduction

Conduction is a microscopic phenomenon. Here, more energetic particles of a substance transfer their energy to their less energetic neighbours. Conduction can occur in a solid, liquid or gas. In a solid, transfer of energy occurs by lattice vibrations and/or free electrons. In a liquid or gas, the transfer of energy occurs by collisions and diffusion of molecules. It should be noted that in solids energy transfer occurs only by conduction whereas in liquids and gases other modes of energy transfer are also possible. Consider, for example, a copper rod, well insulated on its surface, held with its one end inside a furnace at a high temperature; its other end is open to atmosphere at room temperature. It can easily be observed that after some time, the end open to atmosphere will get warmer and reach a temperature higher than ambient. We say that heat is transferred along the rod by conduction.

Governing rate equation for conduction is given by Fourier's law. This is an empirical law based on experimental observations of Biot, but formulated by the French mathematical physicist, Fourier in 1822. It states that *the rate of heat flow by conduction in a given direction is proportional to the area normal to the direction of heat flow and to the gradient of temperature in that direction.*

Referring to Fig. 1.1, for heat flow in the X-direction, Fourier's law states,

FIGURE 1.1 Fourier's law

$$Q_x = -kA\frac{dT}{dx}, W \qquad ...(1.1)$$

i.e. $\qquad Q_x = -kA\,(T_2 - T_1)/L \qquad$ *(For a plane slab, in steady state)*

$$= kA\,(T_1 - T_2)/L$$

or, $\qquad q_x = -k\frac{dT}{dx}, W/m^2 \qquad ...(1.2)$

i.e. $\qquad q_x = k\,(T_1 - T_2)/L$

Here, Q_x is the rate of heat transfer in the positive X-direction, A is the area normal to direction of heat flow, dT/dx is the temperature gradient in the X-direction and k is a proportionality constant. Note that if the heat flow has to be in the positive X-direction, the temperature must go on decreasing in the X direction, i.e. the temperature will decrease as X increases which means that the temperature gradient is negative; therefore, we insert a negative sign in Eqs. 1.1 and 1.2 to make the heat flow positive in the positive X-direction. q_x in Eq. 1.2 is known as heat flux, which is nothing but the heat flow rate per unit area.

The proportionality constant k in Eqs. 1.1 and 1.2 is known as "thermal conductivity", a property dependent on the material. Thermal conductivity of materials varies over a wide range, by about 4 to 5 orders of magnitude. For example, at 20°C, thermal conductivity of air is 0.022 W/(mC), of water 0.51 W/(mC), that of asbestos 0.095 W/(mC), that of stainless steel 19.3 W/(mC) and that of pure silver, about 407 W/(mC). Thermal conductivity, essentially depends upon the material structure (i.e. crystalline or amorphous), density of material, moisture content, pressure and temperature of operation.

We will study more about thermal conductivity in the next chapter.

Example 1.1. Asbestos layer of 10 mm thickness ($k = 0.116$ W/mK) is used as insulation over a boiler wall. Consider an area of 0.5 m² and find out the rate of heat flow as well as the heat flux over this area if the temperatures on either side of the insulation are 300°C and 30°C.

Solution. See Fig. Example 1.1(a). Here, dT/dx is linear i.e. the temperature gradient is linear.

Heat flux q is determined from Eq. 1.2

$$q_x = -k\frac{dT}{dx}, m^2$$
$$= -0.116 \times (30 - 300)/0.01$$
$$= 3132 \text{ W/m}^2$$

Rate of heat flow Q is given by,
$$Q = \text{Heat flux }(q) \times \text{Area}$$
$$= 3132 \times 0.5$$
$$= 1566 \text{ W}.$$

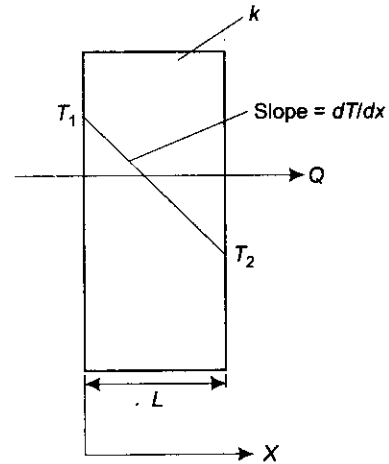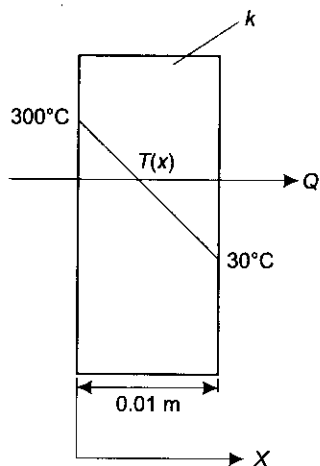**FIGURE** Example 1.1(a)

Now, let us demonstrate working out this problem in Mathcad. A sample worksheet from Mathcad calculation is shown below. Explanatory notes are included; please read them carefully.

First data values are entered. Here, you assign the values of variables, L, k, A, etc. as shown below. Note the assignment symbol, i.e., :=

Data:

$L := 0.01$ m $\quad k := 0.116$ W/(mK) $\quad A := 0.5$ m^2
$T_1 := 300°C \quad T_2 := 30°C$

Solution:

For asbestos:

$$q := k \cdot \frac{(T_1 - T_2)}{L}$$ i.e., $q = 3.132 \times 10^3$ W/m^2

$$Q := q \cdot A$$ i.e., $Q = 1.566 \times 10^3$ W.

Note: This was a simple problem. Now, suppose, you want to see the effect of using another insulation, say, glasswool with $k = 0.038$ W/(mC). Then, re-enter only this data for k and copy the equations again; immediately, you will see that the answers are updated with this new value of k. (There is no need to re-enter the other data again). See below,

For glasswool:

$$k := 0.039 \text{ W/(mK)}$$

k value for glasswool is re-entered. In subsequent calculations, this value of k will be used. Rest of the data values will be as entered earlier.

$$q := k \cdot \frac{(T_1 - T_2)}{L}$$ i.e., $q = 1.053 \times 10^3$ W/m^2

$$Q := q \cdot A$$ i.e., $Q = 526.5$ W.

Here is another example of the versatality of Mathcad: Suppose, we want to use asbestos as the insulation, and one surface maintained at 30°C, but the other surface temperature is varied from 300°C to 350°C, say in steps of 10°C; we wish to compute the corresponding values for q and Q.

For asbestos:

$$k := 0.116 \text{ W/(mC)}$$

re-enter this value to update. Mathcad uses the latest value of variable entered before doing calculation.

$T_1 := 300, 310, ..., 350$ *(Define T_1 as a range variable from 300 to 350 in increments of 10)*

$$q(T_1) := \frac{(T_1 - T_2)}{L}$$ *(Define q as a function of T_1 only, since we are keeping other parameters constant)*

$$Q(T_1) := q(T_1) \cdot A$$ *(Define Q as a function of T_1, since we are keeping other parameters as constant)*

Now, simply type $T = .$ Immediately, a table is produced showing different values of T_1 as shown below. Next, enter $q(T_1)$ and $Q(T_1)$, each followed by ' = ' mark, and tables showing the computed values of q and Q at respective values of T_1 are produced immediately. Arrange them side by side for easy readability:

T_1	$q(T_1)$	$Q(T_1)$
300	2.7×10^4	1.35×10^4
310	2.8×10^4	1.4×10^4
320	2.9×10^4	1.45×10^4
330	3×10^4	1.5×10^4
340	3.1×10^4	1.55×10^4
350	3.2×10^4	1.6×10^4

Graphing. Graphing—the function is very easy in Mathcad. In the graph pallete, press the x-y graph pallete; a blank x-y graph is presented with place holders for x-axis and y-axis variables; just fill them up and the graph is drawn automatically. You can easily add title, grid lines, legend, x-axis and y-axis titles, etc. (See Fig. Example 1.1(b)).
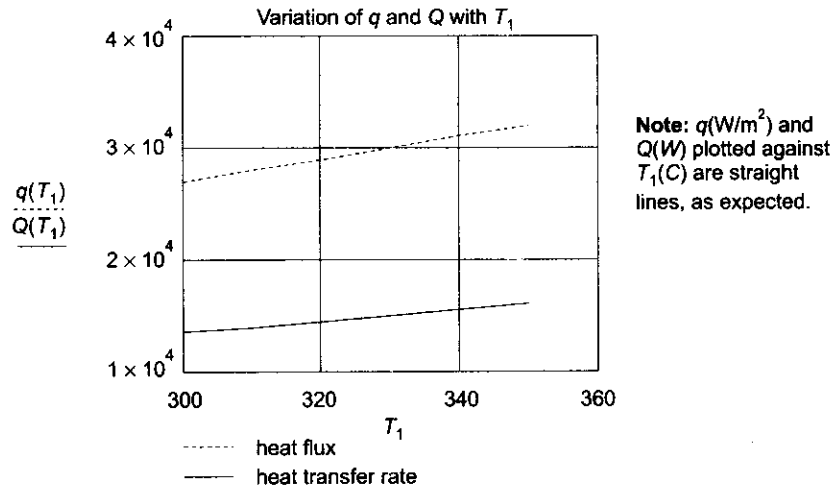
Variation of q and Q with T_1

Note: $q(W/m^2)$ and $Q(W)$ plotted against $T_1(C)$ are straight lines, as expected.

------ heat flux

——— heat transfer rate

FIGURE Example 1.1(b)

1.6.2 Convection

Convection is a macroscopic phenomenon. It occurs only in fluids. When a fluid flows over a body that is at different temperature than itself, heat transfer occurs by convection; the direction of heat transfer, of course, depends on the relative magnitude of the temperatures of the fluid and the surface. In addition, if two fluids at different temperatures are mixed together, heat transfer occurs by convection. Boiling and condensation also involve convective heat transfer, but with phase change.

In convection, the fluid particles themselves move and thus carry energy from a high temperature level to a low temperature level. As an example, consider a hot copper plate held hanging in air. The air layer in the immediate vicinity of the plate gets heated up, its density decreases (since the room air pressure is constant) and therefore rises up, thus carrying away heat with it; the cooler air takes the place of the displaced hot air, gets heated, rises up and this process continues till the plate attains equilibrium with room temperature.

In case of convection, fluid motion may occur by density differences caused by temperature differences, as mentioned in the above example. Such a case is known as *natural* (or *free*) *convection*. When fluid motion is caused by an external agency such as a pump, fan or atmospheric winds, that case is known as *forced convection*. One can intuitively feel that heat transfer in the case of forced convection is higher as compared to free convection.

In the case of convective heat transfer, determining the amount of heat transfer analytically is a little complicated since fluid motion is involved and the equations of fluid flow have to be coupled to the equation of energy.

Governing rate equation for convection is given by Newton's Law of Cooling. Fig. 1.2 shows a situation of natural convection.

Here, a flat plate at a surface temperature of T_s is held vertically; the ambient is at a temperature T_f. Then, the rate of heat transfer is given by Newton's law as follows,

$$Q = hA(T_s - T_f), W \qquad ...(1.3)$$

or,

$$q = h(T_s - T_f), W/m^2 \qquad ...(1.4)$$

where, T_s is the surface temperature (°C) , T_f is the fluid temperature (°C), A is the surface area (m²) exposed to the fluid, Q is the rate of heat transfer (W) from the surface to the fluid, q is the heat flux (W/m²) and h is coefficient of heat transfer for convection.
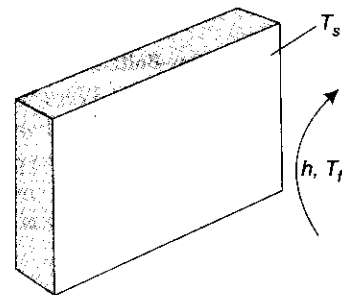

FIGURE 1.2 Newton's Law of Cooling for convection

Of course, if the fluid temperature is higher than the surface temperature, heat will be transferred from the fluid to the surface and in that case, the heat transfer rate is given by,

$$Q = hA(T_f - T_s), \text{W}$$

Here, a few words about h are appropriate. The convective heat transfer coefficient, h, is not a property of the surface material nor that of the fluid. Instead, h is a complicated function of the type of flow (i.e. whether the flow is laminar or turbulent), geometry and orientation of the body, fluid properties (such as specific heat, thermal conductivity, viscosity), average temperature and the position along the surface of the body. In normal practice, even though h varies along the length of the body, it is customary to take an average (or mean) value of h over the entire body, h_m, and use it in Eq. 1.3 to calculate the total heat transfer rate.

Note that Eq. 1.3 does not give any insight into the nature of h and should therefore be considered only as a definition of h.

Typical values of convective heat transfer coefficient, h, for a few situations are given in Table 1.1.

1.6.3 Radiation

All bodies above the temperature of 0 K emit radiation. There are two theories of radiation, i.e. (i) Maxwell's wave theory which states that radiation is emitted as electromagnetic waves, and (ii) Planck's corpuscular theory which states that radiation is emitted in discrete quanta or packets of energy. In practice both these theories are used. Electromagnetic waves travel at the speed of light and generally obey all laws of light. Radiation is emitted over all the wavelengths. However, the radiation emitted over the wavelength range of 0.1 μm to 100 μm is known as *thermal radiation* since radiation in this particular range gets converted to heat when absorbed by a body. Higher the temperature, smaller the wavelength of radiation emitted and deeper its penetration through a body.

TABLE 1.1 Typical values of convective heat transfer coefficient, h

Situation	h, W/(m²K)
Air (1 bar, free convection)	6 – 30
Air (1 bar, forced convection)	10 – 200
Water (free convection)	500 – 1000
Water (forced convection)	600 – 8000
Boiling water	2500 – 100000
Condensing steam	2500 – 70000

Thermal radiation is a volume phenomenon, i.e. the radiation is the result of excitation of all the particles of a body. However, the radiation travels to the surface and is then emitted from the surface.

When radiation falls on a body, it may be attenuated within a short distance from the surface (of the order of a few angstroms), or get reflected from the surface or just pass through the body. One or more of these phenomena may occur simultaneously. In vacuum, radiation propagates without any attenuation. For practical purposes, atmospheric air is considered to be transparent to thermal radiation.

Governing rate equation for emission of radiation flux from a body is given by the Stefan–Boltzmann law:

$$E_b = \sigma T^4, \text{W/m}^2 \qquad \qquad ...(1.5)$$

where, σ = Stefan–Boltzmann constant

$\qquad$ = 5.6697×10^{-8} W/(m²K⁴)

$\qquad$ T = temperature in Kelvin

$\qquad$ E_b = black body emissive power.

Note that Eq. 1.5 defines the emissive power of a *black body*, i.e. an ideal emitter.

Radiation flux emitted by a *real body* is less than that of the black body and is given by,

$$E = \varepsilon E_b = \varepsilon \sigma T^4 \text{ W/m}^2 \qquad \qquad ...(1.6)$$

where, ε is known as Emissivity, lies between zero and unity. Emissivity depends on the surface material, surface finish, temperature and the wavelength of radiation.

By definition, a black body is also an ideal absorber, i.e. it absorbs all the radiation falling on it. However, a real body absorbs only a part of the radiation falling on it.

Hence, we can write,

$$q_{abs} = \alpha\, q_{inc} \; \text{W/m}^2 \qquad\qquad ...(1.7)$$

where, α is absorptivity and lies between zero and unity.

In general, α and ε are different, but for practical purposes, we assume them to be equal to each other.

Radiation exchange between bodies. Consider the case of two surfaces at different temperatures T_1 and T_2 facing each other, separated by a medium like air, which is transparent to radiation. Surface 1 emits radiation at T_1 and surface 2 emits radiation at T_2; there will be a net radiation heat transfer between the two surfaces depending on the magnitudes of T_1 and T_2, emissivities of the surfaces, relative orientation of the surfaces and the distance between them. In general, this calculation is complicated.

However, consider the special case of a small body (A_1, T_1, ε_1) surrounded by a large enclosure (A_2, T_2) as shown in Fig. 1.3.
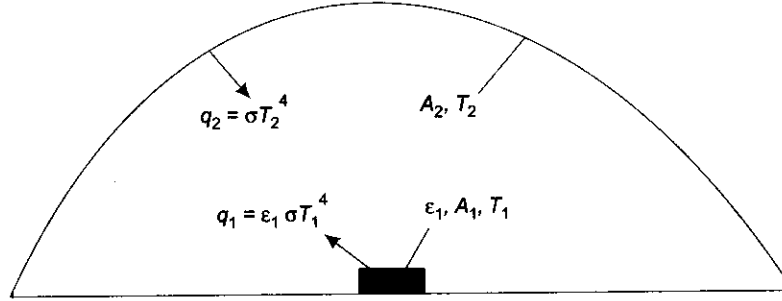

FIGURE 1.3 Radiation from a small surface to a very large surrounding

Let $A_1 << A_2$ and $T_1 > T_2$. Also, the large enclosure can be approximated as a black body with respect to the small surface A_1. Then,

Radiation energy emitted by surface $A_1 = A_1\, \varepsilon_1\, \sigma T_1^4$

Radiation energy flux emitted by black surface $A_2 = \sigma T_2^4$

Out of this energy falling on it, the energy absorbed by surface $A_1 = \alpha_1 A_1\, \sigma T_2^4$

Therefore, net radiation energy leaving the surface A_1 is given by

$$Q_1 = A_1\, \varepsilon_1\, \sigma T_1^4 - A_1\, \alpha_1\, \sigma T_2^4$$

For $\alpha_1 = \varepsilon_1$, we get

$$Q_1 = A_1\, \varepsilon_1\, \sigma(T_1^4 - T_2^4), \text{W} \qquad\qquad ...(1.8)$$

[Note that T_1 and T_2 must be expressed in Kelvin.]

If Q_1 is positive, heat is lost from the surface and if Q_1 is negative, heat is gained by the surface.

Consider, the *case of two finite surfaces* A_1 *and* A_2 facing each other, as shown in Fig. 1.4.

Let the temperatures be T_1 and T_2 and the emissivities ε_1 and ε_2, respectively. Assuming that radiation exchange occurs only between the two surfaces, the net radiation exchange between them is given by

$$Q_1 = F_1 A_1\, \sigma(T_1^4 - T_2^4), \text{W} \qquad\qquad ...(1.9)$$

where, F_1 is known as shape factor or view factor, which includes the effect of orientation, emissivities and the distance between the surfaces. So, determination of F_1 becomes important and we will study about the analysis of such problems in the chapter on radiation.

Radiation heat transfer coefficient. We define a radiation heat transfer coefficient, h_r. This is particularly useful in cases where convection and radiation occur simultaneously. Analogous to convection, we write,

$$q_1 = h_r\,(T_1 - T_2), \text{W/m}^2, \qquad\qquad \textit{(where } h_r \textit{ is the radiation heat transfer}$$
$$\textit{coefficient)}$$

Considering Eq. 1.8, we can write,

$$Q_1 = A_1\, \varepsilon_1\, \sigma(T_1^4 - T_2^4) = A_1\, h_r\,(T_1 - T_2)$$

i.e.
$$h_r = \varepsilon_1\, \sigma(T_1^2 - T_2^2)\,(T_1 + T_2), \text{W/(m}^2\text{K)} \qquad\qquad ...(1.10)$$
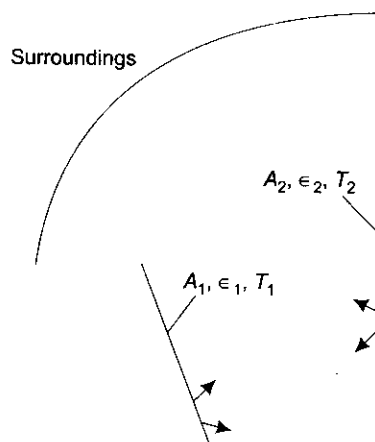
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FIGURE 1.4   Radiation heat exchange between two finite areas

1.6.4   Combined Heat Transfer Mechanism

So far, we dealt with the three modes of heat transfer, namely, conduction, convection and radiation separately, for the sake of analysis. However, in practice, one or more of these modes of heat transfer may occur simultaneously. For example, if a hot casting is removed from its mould and kept open in a room, it will lose heat to the surroundings by convection as well as radiation. Similarly, a roof heated up by Sun will lose heat to the surroundings by convection and radiation. In a heat exchanger, if hot and cold fluids are flowing on either side of the separating wall, the heat transfer involves convection on either side of the wall and conduction through the wall. In all these cases of combined heat transfer, rate equations for the respective modes of heat transfer and the First law of thermodynamics will be used to solve the problems. Example 1.2 will make the procedure clear.

Example 1.2.   A small metallic sphere of emissivity 0.9 loses heat to the surroundings at a rate of 450 W/m^2 by radiation and convection. If the ambient temperature is 300 K and the convective heat transfer coefficient between the sphere and ambient air is 15 W/m^2K, find out the surface temperature of the sphere.
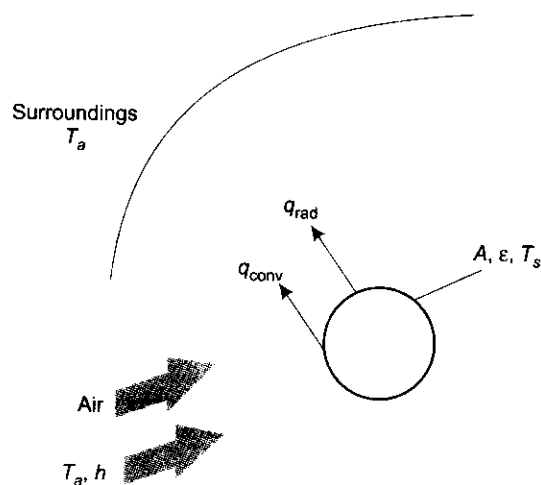
Solution.

FIGURE   Example 1.2

Let the surface area of the sphere be A and surface temperature, T_s. Ambient temperature, $T_a = 300$ K.

Writing an energy balance (i.e. applying the First law):

Total heat lost = heat lost by convection + heat lost by radiation

i.e. $450 \times A = hA\ (T_s - T_a) + A\ \varepsilon\ \sigma(T_s^4 - T_a^4)$

i.e. $450 = 15 \times (T_s - 300) + 0.9 \times 5.67 \times 10^{-8} \times (T_s^4 - 300^4)$

i.e. $15\ T_s + 5.103 \times 10^{-8} \times T_s^4 - 5363.343 = 0$

Solving by trial and error, we get $T_s = 321.3$ K.

[Note: It is important that T_s and T_a are expressed in Kelvin.]

Comment: Many times, combined effect of convection and radiation in a given situation is accounted for by specifying a combined heat transfer coefficient, h_{comb}.

Then, total heat transferred is given by: $Q_{tot} = h_{comb}\ A(T_s - T_a)$.

We shall solve this problem now in Mathcad. Our purpose here is to show the ease with which trial and error solution is done in Mathcad using the 'solve block'.

Let surface area of the sphere be A, heat transfer flux to surroundings, q. Also, surface temperature is T_s and ambient temperature T_a. Heat transfer coefficient for convection is h. Then, from energy balance,

$$q.A = h.A.(T_s - T_a) + A.\varepsilon.\sigma.(T_s^4 - T_a^4)$$

i.e. $$q = h.(T_s - T_a) + \varepsilon.\sigma.(T_s^4 - T_a^4)$$

Data:

$q := 450$ W/m^2 $h := 15$ W/(m^{2}K) $T_a := 300$ K $\varepsilon := 0.9$ $\sigma := 5.67 \cdot 10^{-8}$ W/(m^{2}K^4).

Now, T_s is the unknown. We will apply the energy balance as mentioned above and solve for T_s. However, since a 4th order T_s term is there, we will have to solve it by trial and error.

We use the 'solve block' of Mathcad. Here, firstly, we assume a trial value for the unknown quantity, i.e. for T_s. Let us assume say, $T_s = 400$ K. Then, we write the solve block, starting with Given as shown below. Below Given we write the constraint condition, namely, the energy balance equation in our case. End the solve block by typing Find $(T'_s) =$ and value of T_s appears immediately.

$T_s = 400$ K   (Trial value of T_s)

Given

$$450 = h \cdot (T_s - T_a) + \varepsilon \cdot \sigma \cdot (T_s^4 - T_a^4)$$ (Constraint equation)

Find (T_s) = 321.3 K.

Example 1.3.   Electronic power devices are mounted to a heat sink having an exposed surface area of 0.045 m^2 and an emissivity of 0.8. When the devices dissipate a total power of 20 W and air and surroundings are at 27°C, the average sink temperature is 42°C. What average temperature will the heat sink reach when the devices dissipate 30 W for the same environmental conditions?

Solution.   Let us solve this problem in Mathcad:

Let, the surface area of the heat sink be A, and heat dissipated to surroundings, Q. Also, surface temperature is T_s and surrounding temperature and air temperature T_a. Heat transfer coefficient for convection is h. Then, from energy balance:
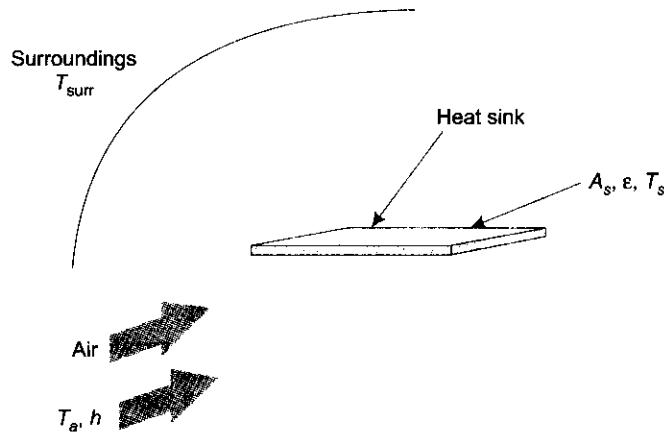

Surroundings
T_{surr}

Heat sink

A_s, ε, T_s

Air

T_a, h

FIGURE   Example 1.3

Heat dissipated by devices = heat lost by sink by convection + heat lost by radiation to surroundings

i.e. $$Q = h.A.(T_s - T_a) + A \cdot \varepsilon \cdot \sigma (T_s^4 - T_a^4)$$

[Important: Express all temperature in Kelvin since radiation calculations are involved.]

Data:

Case 1: h has to be calculated first using the energy balance; all other quantities are known.

$Q := 20$ W $\quad A := 0.045$ m^2 $\quad T_s := 42 + 273$ K $\quad T_a := 27 + 273$ K $\quad T_{surr} := 27 + 273$ K

$\varepsilon := 0.8$ $\quad \sigma := 5.67 \times 10^{-8}$ W/m^{2}K^4

From the energy balance:

$$h = \frac{Q - \sigma \cdot \varepsilon \cdot A \cdot (T_s^4 - T_{surr}^4)}{A \cdot (T_s - T_a)}$$

i.e. $\quad h = 24.351$ W/m^{2}K.

Now, for case 2: If heat dissipated is changed to 30 Watt, what will be the new equilibrium temperature attained by the sink surface? Other conditions remain the same, i.e. now, h value is known, but T_s has to be calculated.

We will apply the energy balance as mentioned above and solve for T_s. However, since a 4th order T_s term is there, we will have to solve it by trial and error.

We use the solve block of Mathcad. Here, firstly, we assume a trial value for the unknown quantity, i.e. for T_s. Let us assume, say, $T_s = 450$ K. Then we write the solve block, starting with Given as shown below. Below Given, we write the constraint condition, namely, the energy balance equation in our case. End the solve block by typing Find(T_s)= and value of T_s appears immediately.

$Q := 30$ W $\qquad\qquad$ *(It is necessary to update the value of Q, since the value will be used in energy balance below in solve block)*

$T_s := 450$ K

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ *(Trial value)*

Given

$$Q = h \cdot A \cdot (T_s - T_a) + \varepsilon \cdot \sigma \cdot A \cdot (T_s^4 - T_{surr}^4)$$

Find $(T_s) = 322.353$ K

That is, the new equilibrirm temperature of the heat sink surface is 322.353 K when the amount of heat dissipated changes to 30 Watt.

1.7 Steady and Unsteady Heat Transfer

In general, temperature distribution within a body depends on position as well as time, i.e. $T = T(x, y, z, \tau)$. When the temperature depends only on spatial coordinates and is independent of time coordinate, we call it *steady state heat transfer*, i.e. $T = T(x, y, z)$ and $T \neq T(\tau)$. However, if the temperature also depends on time coordinate in addition to the spatial coordinates, then we call it *unsteady state heat transfer*, i.e. $T = T(x, y, z, \tau)$. For example, a heat exchanger, when just started, has its wall temperature changing with both position and time, i.e. it undergoes unsteady state heat transfer; after sufficient time elapses, it reaches steady state, i.e. temperature becomes independent of time and is a function of only position. Similarly, start up of a boiler, quenching of a billet, etc., are examples of unsteady state heat transfer. In unsteady state heat transfer, internal energy of the system changes.

Changing of temperature with time can also happen in a cyclic manner, i.e. the same temperature occurs at the same position at definite intervals of time; for example, variation of temperature at a location on earth over a 24-hour cycle, variation of temperature on the cylinder head of an internal combustion engine, etc. This is known as *periodic or quasi-steady state* heat transfer process; in such a case, rate of heat flow and rate of energy storage undergo periodic variation.

We will study more about this topic of unsteady state heat transfer in the chapter on transient conduction.

1.8 Heat Transfer in Boiling and Condensation

In boiling and condensation, there is a phase change involved during heat transfer. During melting, a solid absorbs heat and gets converted to liquid; during boiling, the liquid absorbs heat and gets converted to vapour. Reverse processes occur in solidification and condensation, respectively. In all these cases of heat transfer with phase change, temperature remains constant during the process. Many practical applications of heat transfer with boiling and condensation can be cited: different types of chemical process plants, cryogenic separation of gases, distillation

columns, condensers, reboilers, refrigeration and air conditioning applications etc. Heat transfer in boiling and condensation is characterised by very high values of heat transfer coefficients and should therefore be preferable from the heat transfer point of view. Further, practically isothermal conditions occurring during these processes makes them thermodynamically desirable. However, since the mechanism of boiling and condensation are not amenable to straightforward mathematical treatment, mostly we have to resort to empirical relations to calculate the heat flux. Further, if there is a flow, we would rather like to avoid boiling/condensation and the resulting two-phase flow since precise calculation of pressure drops in such cases is difficult. In the chapter on boiling and condensation, we shall present many useful empirical relations for heat transfer coefficient and heat flux.

1.9 Mass Transfer

Mass transfer is defined as *mass in transit caused as a result of species concentration difference in a mixture.* Just as the temperature difference is the driving potential in case of heat transfer, concentration difference is the driving potential in case of mass transfer.

In a stationary medium, mass transfer occurs purely by diffusion from a plane of high species concentration to a plane of low species concentration. This is analogous to heat transfer by diffusion in case of conduction heat transfer.

Governing rate equation for diffusion mass transfer is given by Fick's law which states that for a binary mixture of species B and C, the diffusion mass flux of the species B is given by,

$$N_b = \frac{m_b}{A} = -D_{bc}\frac{dC_b}{dx} \qquad \qquad ...(1.11)$$

where, $N_b = m_b/A$ = Mass flux of species B, kg/(sm^2)

A = Area through which mass transfer occurs, m^2

C_b = Concentration of species B, kg/m^3

dC_b/dx = Concentration gradient

D_{bc} = Diffusion coefficient or mass diffusivity, m^2/s

Note the similarity between Fick's law for mass diffusion and Fourier's law for heat conduction.

Mass transfer occurs by diffusion as well as convection when a fluid flows over a surface and if there is a concentration difference of a given species in the fluid and the surface. Relations for convective mass transfer with relatively low mass concentration levels in the fluid, are similar to those of convective heat transfer, i.e. an analogy exists between convective heat transfer and convective mass transfer.

There are several practical applications of principles of mass transfer; absorption, desorption, distillation, solvent extraction, drying, humidification, sublimation, etc. In many cases, it is interesting to note that heat and mass transfer occur simultaneously. We will give an introduction to diffusion and convective mass transfer in the chapter on mass transfer.

1.10 Summary

In this chapter, we took a bird's eye view of the three important modes of heat transfer, namely, conduction, convection and radiation. We also mentioned about the topics of heat transfer in boiling and condensation and mass transfer. We studied that there are three rate equations that govern the three modes of heat transfer, namely, Fourier's law for conduction, Newton's law of cooling for convection and Stefan–Boltzmann law for radiation. We briefly mentioned about the difference between the science of heat transfer and thermodynamics, steady state and unsteady state heat transfer and the fundamental laws applied while solving heat transfer problems. Few application areas of heat transfer and mass transfer were listed.

In the subsequent chapters, we will study in detail about the different modes of heat transfer as well as mass transfer and their applications.

Questions

1. In what way is the science of heat transfer different from thermodynamics?
2. Explain with examples the three modes of heat transfer.
3. Explain the respective rate equations governing conduction, convection and radiation and mass diffusion.
4. How is boiling heat transfer different from other modes of heat transfer?
5. Differentiate between steady and unsteady state heat transfer.
6. Mention a few industrial applications of mass transfer.

Problems

1. Determine the heat transfer rate per square metre of surface of a cork board, 3 cm thick, when a temperature difference of 75°C is applied across the board. Take the value of thermal conductivity (k) of cork board as 0.04 W/(mC).

2. What thickness of glasswool (k = 0.038 W/mC) should be used to limit the heat leak rate to 50 W/m^2 when the temperature difference across the layer is 50°C?

3. A fluid at 10°C flows over a 2 cm OD, 2 m long tube whose surface is maintained at 150°C. If the heat transfer coefficient between the fluid and the tube surface is 250 W/(m^{2}C), find out the heat transfer from the tube to the fluid.

4. An 8 cm diameter sphere is heated internally with a 100 W electric heater. Assuming that the sphere dissipates heat only by convection, calculate the surface temperature of the sphere if the convection heat transfer coefficient is 15 W/(m^{2}C). Assume ambient temperature as 25°C.

5. A small plate (5 cm x 5 cm) has its bottom surface insulated and the top surface maintained at 600 K by electric heating. Emissivity of the surface = 0.85. Find out the heat lost by radiation to the surroundings at a temperature of 300 K.

6. A flat plate is heated at a rate of 750 W/m^2 by exposing its one surface to sun and the other surface is insulated. It loses heat by radiation and convection to surroundings at a temperature of 300 K. If the emissivity of the surface is 0.85 and the convective heat transfer coefficient between the plate and the surroundings is 12 W/(m^{2}K), determine the temperature of the plate.

7. A metal casting of size (0.25 m × 0.25 m × 1.0 m high) and at a temperature of 1200 K is removed from its mould and kept standing on its end in a large room. Emissivity of the exposed surfaces is 0.85. Find out the heat lost by radiation and convection if the surrounding is at 300 K and the convective heat transfer coefficient between the casting surfaces and surrounding air is 12 W/(m^{2}K).

Fourier's Law and Its Consequences

2.1 Introduction

While studying the subject of heat transfer, one of our objectives is to calculate the rate of heat transfer. From the second law of thermodynamics, we know that there must be a temperature gradient for heat transfer to occur, i.e. heat flows from a location of high temperature to a location of low temperature. Fourier's law gives the relation between the rate of heat flow and temperature gradient and is therefore considered to be the fundamental law of conduction.

In this chapter, we will first study Fourier's law and the assumptions behind this law. Then, follow two important consequences of Fourier's law; the first one being the definition of thermal conductivity—an important transport property of matter, and the second one being the concept of thermal resistance. We will study about the thermal conductivity of solids, liquids and gases and the variation of this property with temperature. Thermal resistance concept simplifies the solution of many practical problems of steady state heat transfer with no internal heat generation, but involving heat transfer through multiple layers or when different modes of heat transfer occur simultaneously.

2.2 Fourier's Law of Heat Conduction

This is the basic rate equation for heat conduction which gives a relation between the rate of heat transfer and the temperature gradient.

Fourier's law states that *one-dimensional, steady state heat flow rate between two isothermal surfaces is proportional to the temperature gradient causing the heat flow and the area normal to the direction of heat flow.*

Referring to Fig. 2.1, we get,

$$Q \propto A \frac{dT}{dx} \qquad \qquad ...(2.1)$$

$$Q = -kA \frac{dT}{dx} \qquad \qquad ...(2.2)$$

where, Q = heat flow rate in X-direction, W

A = area normal to the direction of heat flow (note this carefully), m^2

dT/dx = temperature gradient, deg./m

k = thermal conductivity, a property of the material, W/(mC) or W/(mK)

This is the differential form of Fourier's equation written for heat transfer in the X-direction.

Negative sign in Eq. 2.2 requires some explanation. We know that heat flows from a location of higher temperature to a location of lower temperature. Referring to Fig. 2.1, if the heat flow rate Q has to occur in the positive X-direction, temperature has to decrease in the positive X-direction, i.e. temperature must decrease as X increases; this
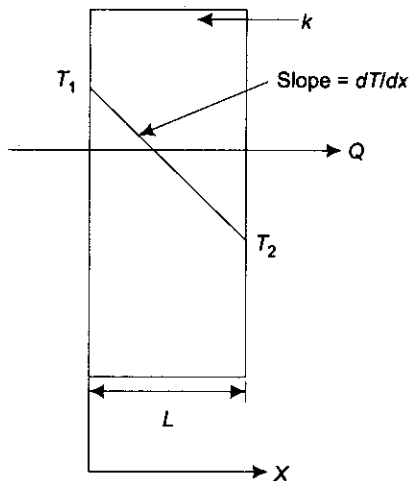
FIGURE 2.1 Fourier's law

means that temperature gradient dT/dx is negative. Since we would like to have the heat flowing in the positive X-direction to be considered as positive, a negative sign is inserted in Eq. 2.2, so that Q becomes positive.

Let us state succinctly the assumptions and other salient points regarding the Fourier's law:

(i) Fourier's law is an empirical law, derived from experimental observations and not from fundamental, theoretical considerations.

(ii) Fourier's law is defined for steady state, one-dimensional heat flow.

(iii) It is assumed that the bounding surfaces between which heat flows are isothermal and that the temperature gradient is constant, i.e. the temperature profile is linear.

(iv) There is no internal heat generation in the material.

(v) The material is homogeneous (i.e. constant density) and isotropic (i.e. thermal conductivity is the same in all directions).

(vi) Fourier's law is applicable to all states of matter, i.e. solid, liquid or gas.

(vii) Fourier's law helps to define thermal conductivity i.e. from Eq. 2.2 we can write, for steady state heat transfer through a slab of thickness L and its two surfaces at constant temperatures of T_1 and T_2, $(T_1 > T_2)$.

$$Q = -kA\ dT/dx$$
$$= -kA\ (T_2 - T_1)/L$$
$$= kA\ (T_1 - T_2)/L$$

We can say that

$$Q = <k> \text{ when } A = 1 \text{ m}^2, dT = 1 \text{ deg., } dx = 1 \text{ m,}$$

i.e. thermal conductivity of a material is numerically equal to the heat flow rate through an area of one m^2 of a slab of thickness 1 m with its two faces maintained at a temperature difference of one degree celcius.

Therefore, the unit of thermal conductivity is obtained from:

$$k = Q\frac{L}{A(T_1 - T_2)}, \text{ W/(mC) or W/(mK)} \qquad \text{...(2.3)}$$

Note that W/(mC) and W/(mK) mean the same thing in Eq. 2.3, $(T_1 - T_2)$ is the temperature difference which is the same whether it is deg.C or deg.K.

2.3 Thermal Conductivity of Materials

We state Fourier's law again:

$$Q = -kA\ dT/dx$$
$$= kA\ (T_1 - T_2)/\text{L}.$$

Here, k is the thermal conductivity, a property of the material. Its units: W/(mC) or W/(mK). Thermal conductivity, essentially depends upon the material structure (i.e. crystalline or amorphous), density of material, moisture content, pressure and temperature of operation.

Thermal conductivity of materials varies over a wide range, by about 4 to 5 orders of magnitude. For example, thermal conductivity of Freon gas is 0.0083 W/(mC) and that of pure silver is about 429 W/(mC) at normal pressure and temperature.

Fig 2.2 shows the range of variation of thermal conductivity of different classes of materials:

Table 2.1 gives values of thermal conductivities for a few materials at room temperature.

2.3.1 Thermal Conductivity of Solids

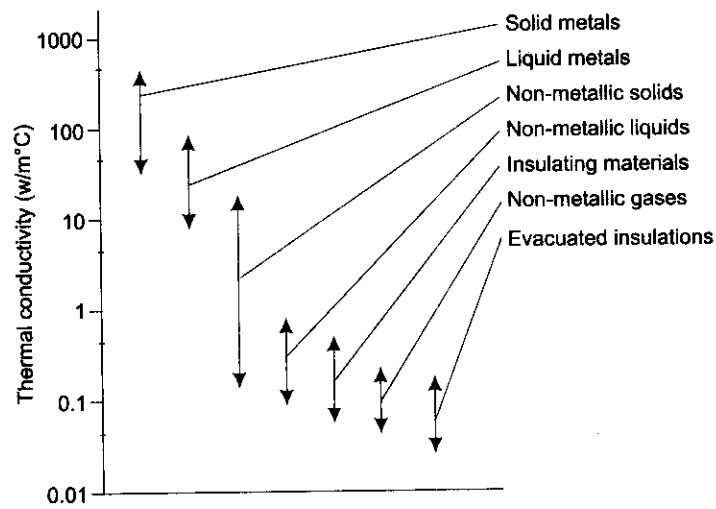Thermal conductivity of solids is made up of two components,

FIGURE 2.2 Range of thermal conductivities of various materials

(i) due to flow of free electrons, and

(ii) due to lattice vibrations.

First effect is known as *electronic conduction* and the second effect is known as *phonon conduction*.

2.3.1.1 Metals and alloys. In case of pure metals and alloys,

(a) There is an abundance of free electrons and the electronic conduction predominates. Since free electrons are also responsible for electrical conduction, it is observed that good electrical conductors are also good thermal conductors, e.g. copper, silver etc.

(b) Any effect which inhibits the flow of free electrons in pure metals reduces the value of thermal conductivity. For example, with a rise in temperature, the lattice vibration increases and this offers a resistance to the flow of electrons and therefore, for pure metals thermal conductivity decreases as temperature increases (uranium and aluminium are exceptions). Fig. 2.3 shows the variation of thermal conductivity with temperature for a few metals.

TABLE 2.1 Thermal conductivity of a few materials at room temperature

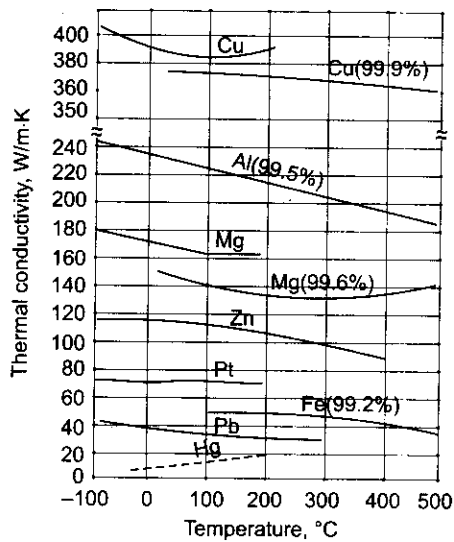| Material	k, W/mC
Diamond	2300
Silver	429
Copper	401
Gold	317
Aluminium	237
Iron	80.2
Mercury (l)	8.54
Glass	0.78
Brick	0.72
Water (l)	0.613
Wood (oak)	0.17
Helium (g)	0.152
Refrigerant-12	0.072
Glass fibre	0.043
Air (g)	0.026

FIGURE 2.3 Variation of thermal conductivity with temperature for a few metals

(c) Alloying decreases the value of thermal conductivity since the *foreign atoms* cause scattering of free electrons, thus impeding their free flow through the material. For example, thermal conductivity of pure copper near about room temperature is 401 W/(mC) while presence of traces of arsenic reduces the value of thermal conductivity to 142 W/(mC).

(d) Heat treatment, mechanical forming and cold working reduce the value of thermal conductivity of pure metals.

(e) Thermal conductivity of alloys generally increases as temperature increases. Fig. 2.4 below confirms this trend for a few alloys.

(f) Since the phenomenon of electron conduction is responsible for both thermal conduction and electrical conduction, it is reasonable to presume that there must be relation between these two quantities. In fact, Weidemann–Franz law gives this relation. This law, based on experimental results, states *the ratio of thermal and electrical conductivities is the same for all metals at the same temperature and this ratio is directly proportional to the absolute temperature of the metal.*

$$\frac{k}{\sigma} \propto T$$

$$\frac{k}{\sigma T} = C \qquad\qquad ...(2.4)$$
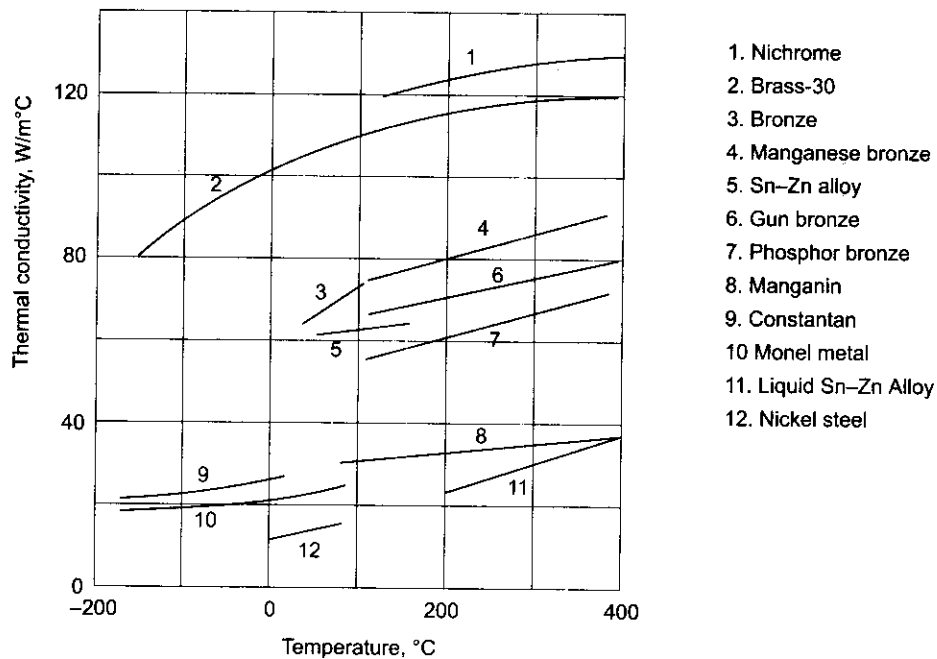

1. Nichrome
2. Brass-30
3. Bronze
4. Manganese bronze
5. Sn–Zn alloy
6. Gun bronze
7. Phosphor bronze
8. Manganin
9. Constantan
10 Monel metal
11. Liquid Sn–Zn Alloy
12. Nickel steel

FIGURE 2.4 Variation of thermal conductivity with temperature for a few alloys

FUNDAMENTALS OF HEAT AND MASS TRANSFER

where k = thermal conductivity of metal, W/(mK)

   σ = electrical conductivity of metal, $(\text{ohm.m})^{-1}$

   C = Lorentz number, a constant for all metals

   $= 2.45 \times 10^{-8}$ W Ohms/K^2

An important practical application of Weidemann–Franz law is to determine the value of thermal conductivity of a metal at a desired temperature, knowing the value of electrical conductivity at the same temperature. Note that it is easier to measure experimentally the value of electrical conductivity than that of thermal conductivity.

2.3.1.2   Non-metallic solids. In case of non-metallic solids,

(a) For dielectrics, there are no free electrons and the thermal conductivity values are much lower than those of metals. For heat insulating materials, general range of values of k are from 0.023 W/(mC) to 2.9 W/(mC). Thermal conductivity increases with temperature for insulating materials as shown in Fig. 2.5.

1. Air
2. Mineral wool
3. Slag wool
4. 85% Magnesia
5. Sovelite
6. Diatomaceous brick
7. Red brick
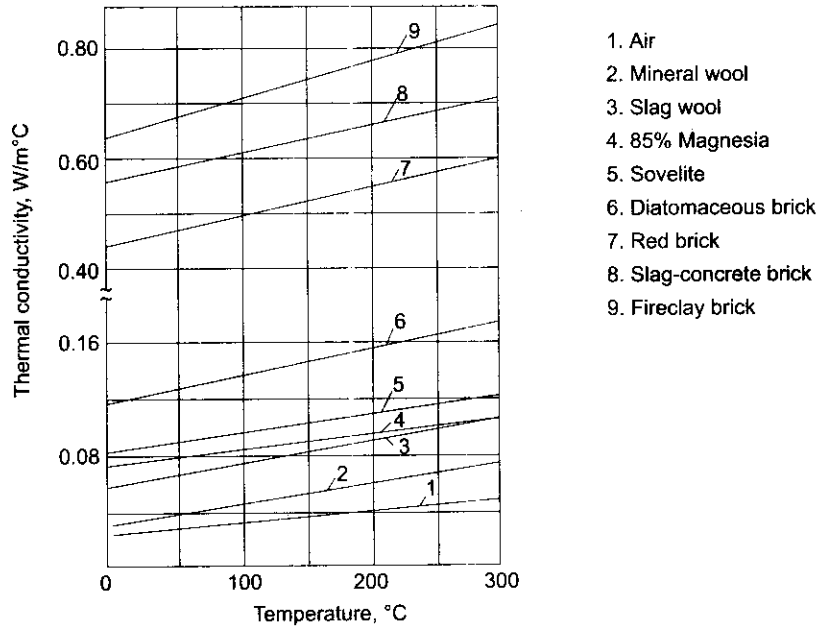8. Slag-concrete brick
9. Fireclay brick

FIGURE 2.5   Variation of thermal conductivity with temperature for insulating materials

(b) For porous heat insulating materials (brick, concrete, asbestos, slag, etc.), thermal conductivity depends greatly on density of the material and the type of gas filling the voids. For example, k of asbestos increases from 0.105 to 0.248 W/(mC) as density increases from 400 to 800 kg/m^3; this is due to the fact that thermal conductivity of air filling the voids is much less than that of the solid material.

(c) Thermal conductivity of porous materials also depends on the moisture content in the material; k of a damp material is much higher than that of the dry material and water taken individually.

(d) Thermal conductivity of granular materials increases with temperature since with increasing temperature, radiation from the granules also comes into picture along with conduction of medium filling the spaces.

(e) *Variation of thermal conductivity of solids with temperature:* In heat transfer calculations, generally we assume k to be constant when the temperature range is small; however, when the temperature range is large, it is necessary to take into account the variation of k with temperature.

Usually, for solids, a linear variation of thermal conductivity with temperature can be assumed without loss of much accuracy.

$$k(T) = k_0(1 + \beta T) \qquad \qquad ...(2.5)$$

where, $k(T)$ = thermal conductivity at desired temperature T, W/(mC)

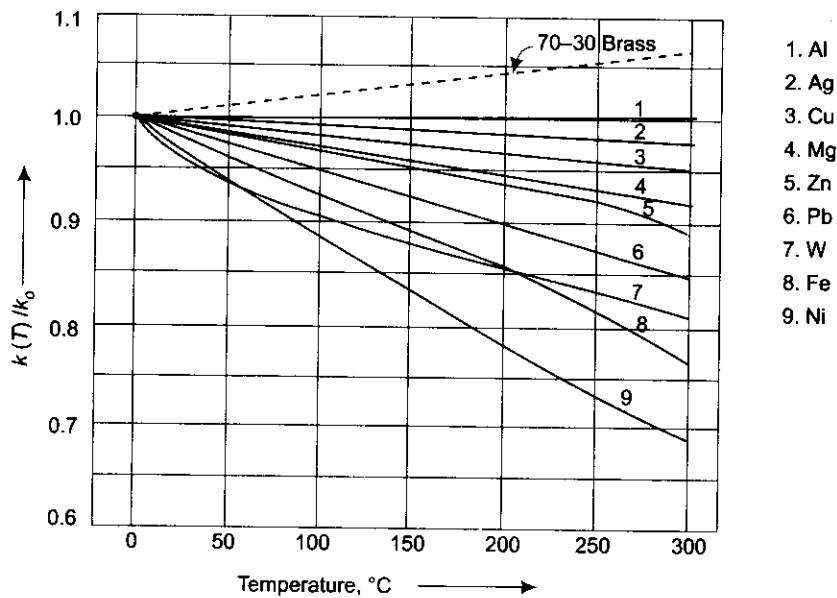 k_0 = thermal conductivity at reference temperature of 0°C, W/(mC)

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FIGURE 2.6 Variation of thermal conductivity with temperature for a few pure metals

TABLE 2.2 Representative values of k_0 and β in Eq. 2.5

Material	k_0 (W/mC)	$\beta \times 10^4$, (1/C)
Metals and alloys		
Aluminium	246.985	− 2.227
Chromium	97.123	− 5.045
Copper	401.5275	− 1.681
Stainless steel	14.695	+ 10.208
Uranium	26.679	+ 8.621
Insulators		
Fireclay brick	0.76	0.895
Red brick	0.56	0.66
Sovelite	0.092	0.12
85% Magnesia	0.08	0.101
Slag wool	0.07	0.101
Mineral wool	0.042	0.07

β = a temperature coefficient, 1/C

T = temperature, °C

Fig. 2.6 shows the variation of k with temperature for a few pure metals. It may be noted that the variation is linear as indicated in Eq. 2.5.

In Eq. 2.5, value of β may be positive or negative. Generally, β is negative for metals (exception being uranium) and positive for insulators and alloys. Table 2.2 gives representative values of k_0 and β for a few materials.

2.3.2 Thermal Conductivity of Liquids

2.3.2.1 Non-metallic liquids. Heat propagation in liquids is considered to be due to elastic oscillations. As per this hypothesis, the thermal conductivity of liquids is given by,

$$k = \frac{A c_p \rho^{\frac{4}{3}}}{M^{\frac{1}{3}}}$$

...(2.6)

where, c_p = specific heat of liquid at constant pressure

      ρ = density of liquid

      M = molecular weight of liquid

      A = constant depending on the velocity of elastic wave propagation in the liquid; it does not depend on nature of liquid, but on temperature.

It is noted that the product $A.c_p$ is nearly constant. As temperature rises, density of a liquid falls and as per Eq. 2.6 the value of thermal conductivity also drops for liquids with constant molecular weights. (i.e. for non-associated or slightly associated liquids). This is generally true as shown in Fig. 2.7.

Notable exceptions are water and glycerin, which are heavily associated liquids. With rising pressure, thermal conductivity of liquids increases. For liquids, k value ranges from 0.07 to 0.7 W/(mC).
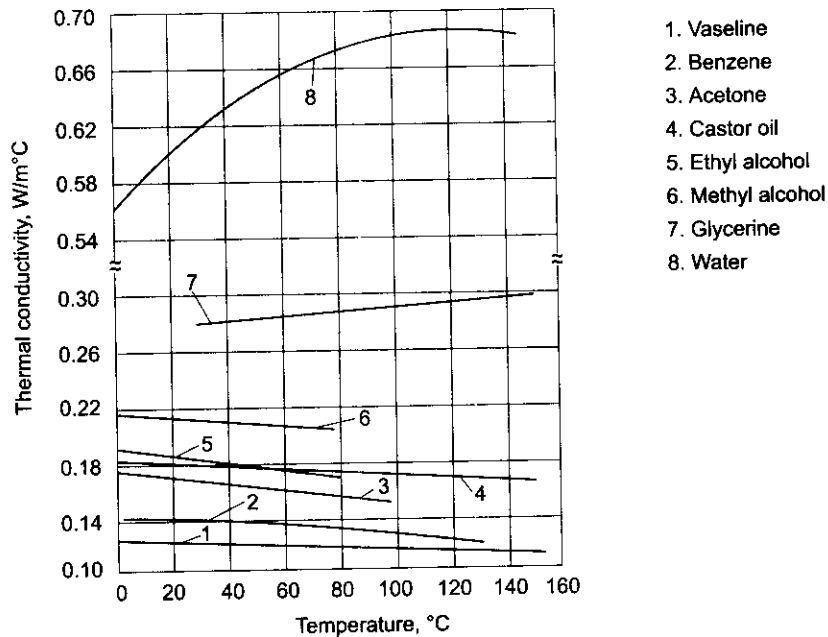

1. Vaseline
2. Benzene
3. Acetone
4. Castor oil
5. Ethyl alcohol
6. Methyl alcohol
7. Glycerine
8. Water

FIGURE 2.7 Thermal conductivity of non-metallic liquids

2.3.2.2 Liquid metals. Liquid metals like sodium, potassium etc. are used in high flux applications as in nuclear power plants where a large amount of heat has to be removed in a small area. Thermal conductivity values of liquid metals are much higher than those for non-metallic liquids. For example, liquid sodium at 644 K has k = 72.3 W/(mK); liquid potassium at 700 K has k = 39.5 W/(mK); and liquid bismuth at 589 K has k = 16.4 W/(mK).

2.3.3 Thermal Conductivity of Gases

(a) Heat transfer by conduction in gases at ordinary pressure and temperature is explained by the Kinetic Theory of Gases. Temperature is a measure of kinetic energy of molecules. Random movement and collision of gas molecules contribute to the transport of kinetic energy, and, therefore, to transport of heat. So, the two quantities that come into picture now are: the mean molecular velocity, V and the mean free path, l. Mean free path is defined as the mean distance travelled by a molecule before it collides with another molecule.

FOURIER'S LAW AND ITS CONSEQUENCES

Thermal conductivity of gases is given by,

$$k = \frac{1}{3} V l c_v \rho \qquad \qquad ...(2.7)$$

where, V = mean molecular velocity

l = mean free path

c_v = specific heat of gas at constant volume

ρ = density

(b) As pressure increases, density ρ increases, but the mean free path l decreases almost by the same proportion and the product l ρ remains almost constant, i.e the thermal conductivity of gases does not vary much with pressure except at very low (less than 20 mmHg) or very high (more than 20,000 bar) pressures.

(c) As to the effect of temperature on thermal conductivity of gases, mean molecular velocity V depends on temperature as follows,

$$V = \sqrt{\frac{3GT}{M}}$$

where G = Universal gas constant = 8314.2 J/kmol K

M = molecular weight of gas

T = absolute temperature of gas, K

i.e. mean molecular velocity varies directly as the square root of absolute temperature and inversely as the square root of molecular weight of a gas. Specific heat, c_v also increases as temperature increases. As a result, thermal conductivity of gases increases as temperature increases.

(d) For the reason stated above, gases with lower molecular weights such as helium and hydrogen have higher values of thermal conductivities (almost by 5 to 10 times) as compared to gases with higher molecular weights such as air.

(e) Generally, thermal conductivity values for gases vary in the range 0.006 to 0.6 W/(mC)

(f) Thermal conductivity of steam and other imperfect gases depend very much on pressure unlike that of perfect gases.

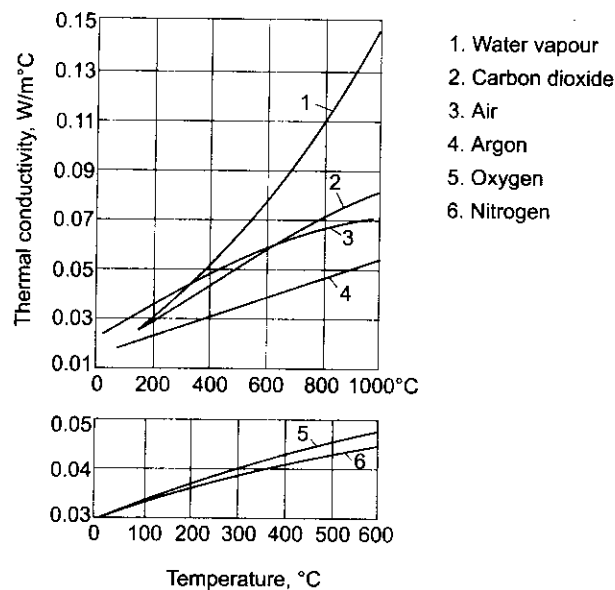Fig. 2.8 and Fig. 2.9 show the variation of k with temperature for a few gases.

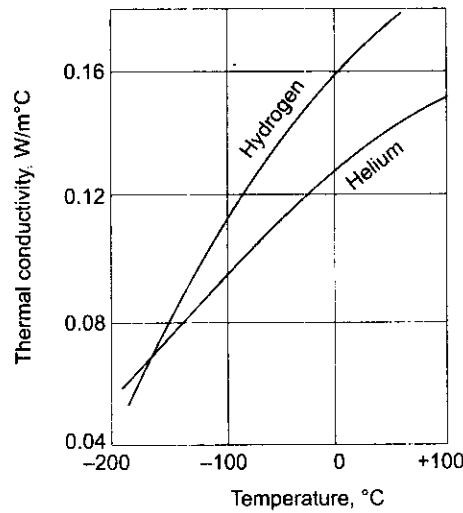FIGURE 2.8   Variation of k with temperature for a few gases

FIGURE 2.9 Variation of k with temperature for hydrogen and helium

2.3.4 Insulation Systems

It is appropriate here to consider insulation systems generally used. In industries where huge amount of thermal energy is dealt with, be it for high temperature or low temperature application, it is necessary to see that the most suitable insulation is adopted. This has become particularly important now, since there is widespread awareness about the *energy crunch* and the *cost of energy*.

Insulation is required for high temperature systems as well as low temperature systems. In high temperature systems, any leakage of heat from boilers, furnaces or piping carrying hot fluids represents an energy loss. Similarly, in low temperature/cryogenic systems, any heat leakage into the low temperature region represents an energy loss since from thermodynamics we know that to pump out a given amount of heat from a low temperature region would need a disproportionately large amount of work to be put in at room temperature.

Insulation systems may be classified as,

   (i) fibrous
   (ii) cellular
   (iii) powder
   (iv) reflective.

Since in non-homogeneous insulation materials, a combination of conduction, convection or radiation is involved, they are characterised by an "effective thermal conductivity". Solid materials have *cells* of spaces formed inside them by *foaming*. There may be air or some other gas inside these voids. Type of gas used affects the property of the material. Obviously, density of these systems plays an important role in determining the effective thermal conductivity. Sometimes, the intervening spaces are evacuated to reduce the convection losses. To get extremely low values of thermal conductivity—of the order of a few $\mu W/(mK)$—multiple layers of highly reflective materials are introduced in between the insulation layers. These are called *superinsulations* and are used in cryogenic and space applications.

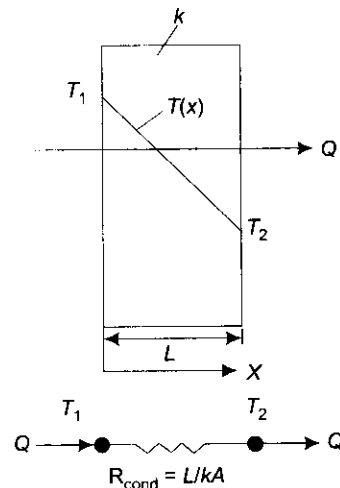Table 2.3 gives details about some of the common insulations used in industry.

FIGURE 2.10 Conduction heat flow through a slab—thermal resistance

TABLE 2.3 Common Insulations used in Industry

Insulation material	Density (kg/m³)	Temperature range, °C	Thermal conductivity (mW/(mC))	Application
Mineral wool blankets	175–290	450–1000	50–130	Vessels, pipes
Mineral fibre blankets	125	Up to 750	37–80	Vessels, pipes
Fibre glass mats	10–50	60–360	30–55	Vessels, tanks
Mineral fibre block	210	Up to 1100	50–130	Boilers and tanks
Calcium silicate, board/block	25–100	230–1000	32–85	Boilers, chimney liners
Fibre glass board	25–100	20–450	32–52	Boilers, tanks, heat exchangers
Fibre glass blanket	10–50	–160 to 230	24–86	Chillers, tanks, vessels
Expanded polystyrene	20–50	–100 to 40	22–25	Chilled vessels
Polyurethane foam	25–50	–180 to –150	16–20	Low temp. piping
Polyurethane foam	32	–170 to 110	16–20	Tanks, vessels (cold/hot)
Fine perlite, evacuated to < 0.001 torr	180	–200 to 30	0.95	Cryogenic service
0.006 mm Al foil + 0.15 mm fibreglass paper, vacuum 10⁻⁵ torr	20 layers/cm	Less than –150	0.037	Cryogenic service
0.0087 mm Al foil + carbon loaded glass-fibre paper, vacuum 10⁻⁵ torr	30 layers/cm	Less than –150	0.014	Cryogenic service

2.4 Concept of Thermal Resistance

2.4.1 Conduction

Consider a slab of thickness L, constant thermal conductivity k, with its left and right faces maintained at temperatures T_1 and T_2. If T_1 is greater than T_2, we know that heat will flow from left to right and the heat flow rate is given by Fourier's law,

$$Q = kA(T_1 - T_2)/L \qquad \qquad ...(2.9)$$

Now, consider this: in a pipe carrying a fluid, the flow occurs under a driving potential of a pressure difference and there is resistance to flow due to pipe friction; in an electrical conductor, flow of electricity occurs under the driving potential of a voltage difference and there is a resistance to the flow of electric current. Similarly, considering Eq. 2.9, we can say that flow of heat Q occurs in the slab by conduction under a driving potential of a temperature difference $(T_1 - T_2)$ and the material offers a *thermal resistance* to the flow of heat. So, we can write Eq. 2.9 as,

$$Q = \frac{T_1 - T_2}{\dfrac{L}{kA}} = \frac{\Delta T}{\dfrac{L}{kA}}$$

$$= \frac{\Delta T}{R_{th}} = \frac{Temperature\ difference}{Thermal\ resistance} \qquad ...(2.10)$$

$R_{th} = L/(kA)$ is known as *Thermal resistance* of the slab for conduction.

It is seen that there is a clear *analogy* between the flow of heat and flow of electricity, as shown below,

Situation	Driving force	Flow	Resistance
Electric circuit	Voltage, ΔV	Current, I	Electric Resistance $R = \Delta V/I$
Thermal circuit	Temp. Difference, ΔT	Heat flow rate, Q	Thermal Resist., $R_{th} = \Delta T/Q$

Fig. 2.10 above shows the thermal circuit for the situation of flow of heat through a plane slab by conduction. For the slab, we write,

$$R_{cond} = L/(kA) \qquad \qquad ...(2.11)$$

Note that units of thermal resistance is (C/W) or, K/W.

2.4.2 Convection

Consider the case of a fluid flowing with a free stream velocity U and free stream temperature T_f, over a heated surface maintained at a temperature T_s. Let the heat transfer coefficient for convection between the surface and the fluid be h. Then, the heat transfer rate from the surface to the fluid is given by Newton's rate equation,

$$Q = hA\,(T_s - T_f)$$

This can be written as,

$$Q = \frac{T_s - T_f}{\dfrac{1}{hA}} = \frac{\Delta T}{\dfrac{1}{hA}} = \frac{Temperature\ difference}{Convective\ thermal\ resistance} \qquad ...(2.12)$$

Again, note the analogy between flow of electricity and the flow of heat (see Fig. 2.11).

So, for heat transfer by convection, we write,

$$R_{conv} = \frac{1}{(hA)} \qquad ...(2.13)$$

Note that the units are (C/W) or (K/W).

2.4.3 Radiation

For the case of heat transfer between two finite surfaces, at temperatures T_1 and T_2 (Kelvin), net radiation heat transfer between them is given by equation,

$$Q_1 = F_1 A_1 \sigma (T_1^4 - T_2^4),\ W$$

FIGURE 2.11 Convection heat transfer— thermal resistance

where, F_1 is known as *shape factor* or *view factor*, which includes the effects of orientation, emissivities and the distance between the surfaces. σ is the Stefan–Boltzmann constant.

Write the above equation in the following form,

$$Q = \frac{T_1 - T_2}{\dfrac{1}{F_1 A_1 \sigma (T_1 + T_2)(T_1^2 + T_2^2)}} = \frac{\Delta T}{R_{rad}} = \frac{Temperature\ difference}{Radiation\ thermal\ resistance} \qquad ...(2.14)$$

Clearly, the radiation thermal resistance may be written as,

$$R_{rad} = \frac{1}{F_1 A_1 \sigma (T_1 + T_2)(T_1^2 + T_2^2)},\ C/W \qquad ...(2.15)$$

2.4.4 Practical Applications of Thermal Resistance Concept

There are two important practical application of the thermal resistance concept:

(i) To analyse the problems where *one or more modes of heat transfer occur simultaneously*. For example, in a heat exchanger plate when a hot fluid flows on one side and a cold fluid on the other side, we have heat transfer occurring at either surface by convection and through the plate itself by conduction. Obviously, the thermal resistances in this case are all in series and the rules of series resistances in an electrical circuit apply, i.e. total thermal resistance is the sum of the three resistances.

i.e. $\qquad \qquad R_{eff} = R_{conv1} + R_{cond} + R_{conv2}$

But in some other cases, the thermal resistances may be in parallel; for example, a heated wall of a furnace may lose heat to ambient by convection as well as radiation, i.e. heat transfer occurs from the wall by these two modes simultaneously in parallel. Then we apply the rule for parallel resistances, i.e. effective resistance is given by,
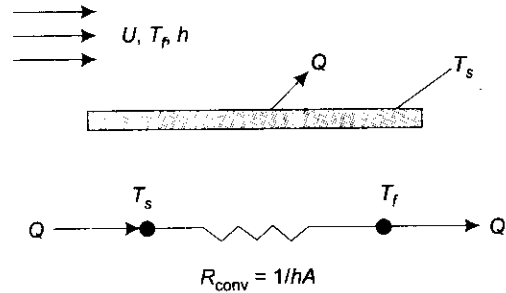
$$\frac{1}{R_{\text{eff}}} = \frac{1}{R_{\text{conv}}} + \frac{1}{R_{\text{rad}}}$$

(ii) To analyse the problems where *multiple layers* of materials of different thermal conductivities are used; e.g. in furnace walls which are lagged with 2 or 3 layers of insulation, insulation of walls of houses in cold weather, lagging of pipes, etc. Since the thermal conductivities and thicknesses of materials used may be different, thermal resistances of individual layers are different and it becomes convenient to use the thermal resistance concept to find out the total resistance and hence the heat flow rate.

2.4.5 Limitations for the Use of Thermal Resistance Concept

Thermal resistance concept can be used only when all the following conditions are satisfied.

(i) One-dimensional conduction

(ii) Steady state conduction

(iii) No internal heat generation.

Note: In this chapter, we have just introduced the concept of thermal resistance. We will study more about this concept and apply it to analyse heat transfer in composite slabs, cylinders and spheres and also to situations where more than one mode of heat transfer exist simultaneously, in Chapter 4. Therein, we shall also solve several numerical problems to illustrate the applications of this concept.

2.5 Thermal Diffusivity (α)

Often, during heat transfer analysis, particularly while dealing with transient conduction problems, we come across a quantity called *Thermal diffusivity*, defined as,

$$\alpha = \frac{k}{\rho c_p}, \quad \frac{m^2}{s} \qquad \qquad \qquad ...(2.16)$$

where, k = thermal conductivity of the material, $W/(mC)$

ρ = density, kg/m^3

c_p = specific heat at constant pressure, $J/(kg.C)$

Note that unit of α is m^2/s.

Let us consider the **physical significance of thermal diffusivity**, α: Thermal conductivity (k) of a material is a transport property and denotes its ability to conduct heat; higher the value of k, better the ability of material *to conduct heat*. The product (ρc_p) is known as *volumetric heat capacity*, has units of $J/(m^3 K)$, and denotes the ability of the material *to store heat*. Higher the value of (ρc_p), larger the heat storage capacity. Generally, solids and liquids which are good storage media have higher volumetric heat capacity ($> 1 \ MJ/m^3 \ K$) as compared to gases (about 1 kJ/m^3 K), which are poor heat storage media. Therefore, thermal diffusivity, i.e. the ratio of k to (ρc_p) gives the relative ability of the material to conduct heat as compared to its ability to store heat. Larger the value of α, faster the propagation of heat into the material. In other words, α represents the ability of the material to respond to changes in the thermal environment; larger the value of α, quicker the material will come into thermal equilibrium with its surroundings. Values of α for materials vary over a wide range. For example, for copper at room temperature, its value is approx. $113 \times 10^{-6} \ m^2/s$, whereas for glass it is about $0.34 \times 10^{-6} \ m^2/s$.

Table 2.4 shows typical values of thermal diffusivity for a few materials.

2.6 Summary

In this chapter, we studied Fourier's law for one-dimensional conduction. This is a very important topic and student must be clear about the assumptions behind this law; particularly, you should note that the area used in applying this law is the area normal to the direction of heat flow. Fourier's law opens the door for further learning about conduction; we will use it immediately in the next chapter to derive the general differential equation for conduction heat transfer. In this chapter, we also studied two important consequences of Fourier's law: firstly, definition of thermal conductivity—an important transport property of material—and, secondly, concept of thermal resistance. We studied in some detail about the thermal conductivity of solids, liquids and gases and the variation of thermal conductivity with temperature. Thermal diffusivity—a significant property while studying transient conduction—was mentioned and its physical significance explained.

TABLE 2.4 Typical values of thermal diffusivity (α) for a few materials at room temperature

Material	$\alpha \times 10^6$, (m^2/s)
Silver	149
Gold	127
Copper	113
Aluminium	97.5
Iron	22.8
Mercury (l)	4.7
Marble	1.2
Ice	1.2
Concrete	0.75
Brick	0.52
Glass	0.34
Glasswool	0.23
Water (l)	0.14
Beef	0.14
Wood (oak)	0.13

In the next chapter, we shall derive the general differential equation for conduction which, when solved, will give the temperature distribution in a material; knowing the temperature distribution, we can easily determine the heat transfer rate by applying the Fourier's law.

Questions

1 State and explain Fourier's law for one-dimensional conduction. What are the underlying assumptions?
2. What are the important consequences of Fourier's law?
3. Define 'thermal conductivity'. What are the factors affecting the thermal conductivity of a material?
4. Write a short note on thermal conductivity of solids, liquids and gases.
5. How does thermal conductivity vary with temperature for metals, alloys and insulators?
6. Name the different insulations used in industry and mention the specific purpose for which each is used.
7. Explain the analogy between flow of heat and flow of electricity.
8. Explain the concept of thermal resistance. What are the practical uses of this concept?
9. What do you understand by the term 'thermal diffusivity'? Explain its physical significance.
10. On a cold, winter morning, the aluminium handle of the front door of your house feels cold to touch as compared to the wooden door frame, even though both were exposed to the same cold environment throughout the night. Explain why?
11. The inner and outer surfaces of a 5 m × 6 m brick wall of thickness 30 cm and thermal conductivity 0.69 W/(mC), are maintained at temperatures of 20°C and 35°C, respectively. Determine the rate of heat transfer through the wall.
12. In an experiment to find out the thermal conductivity of a material, an electric heater is sandwiched between two identical samples, each of size (10 cm × 10 cm) and thickness 0.5 cm, and all the four outer edges are well insulated. At steady state, it is observed that the electric heater draws 35 W of power and the temperature of each sample was 90°C on the inner surface and 82°C on the outer surface. Determine the thermal conductivity of the material at the average temperature.
13. By conduction, 3 kW of energy is transferred through 0.5 m^2 section of a 5 cm thick insulating material of thermal conductivity 0.2 W/(mC). Determine the temperature difference across the layer.

General Differential Equations for Heat Conduction

3.1 Introduction

In heat transfer analysis, one of our objectives is to determine the temperature distribution within the body at any given instant, i.e. the temperature at every point in the body, taken as a continuum. Then, we can calculate the heat transfer rate at any point in a given direction by applying the Fourier's law. Knowledge of temperature distribution is also required in other fields of engineering analysis; e.g. in calculation of thermally induced stresses, thermal expansion , optimum thickness of insulation, etc.

General technique to obtain the temperature distribution over the entire body is to consider a differential control volume within the body and apply the *law of conservation of energy* to this differential control volume. This results in a differential equation. Solution of this differential equation with appropriate initial and boundary conditions gives the temperature field, i.e. the temperature at any point within the body.

In this chapter, first, general differential equations for conduction is derived in Cartesian (i.e. rectangular or x-y-z) coordinates. This is useful to analyse problems of heat transfer in rectangular-shaped bodies such as squares, rectangles, parallellopiped, etc. Next, the general differential equation of conduction is stated in cylindrical and spherical coordinates; these are useful to solve heat transfer problems in cylinders and spheres. Simplifications to the general differential equations for different, possible practical conditions are presented. Typical boundary conditions encountered in practice and the methods to represent them mathematically are explained. A summary of the equations is given at the end of the chapter for ready reference.

3.2 General Differential Equation for Heat Conduction in Cartesian Coordinates

This is also known as **heat diffusion equation** or, simply **heat equation**. Consider a homogeneous body within which there is no bulk motion and heat transfer occurs in this body by conduction. Temperature distribution within the body at any given instant is given by: $T(x, y, z, \tau)$. The coordinate system used in this derivation is given in Fig. 3.1.

Consider a differential volume element $dx.dy.dz$ from within the body as shown. It has six surfaces and each surface may be assumed to be isothermal since the differential element is very small. Further, the body is assumed to be rigid, i.e. negligible work is done on the body by external mechanical forces.

Let us make an energy balance on this differential element. Let us list out the various energy terms involved: first, there is energy conducted into the element; second, there is energy conducted out of the element; third, for generality, let there be energy generated within the element, say, due to joule heating, chemical reaction or nuclear fission, etc. Net heat conducted into the element in conjunction with the heat generated within the element, will
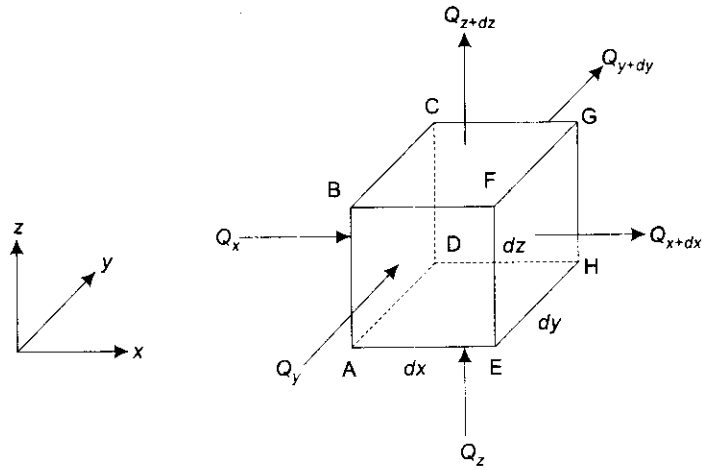
FIGURE 3.1 Nomenclature for derivation of general differential equation for heat conduction in Cartesian coordinates

obviously cause an increase in the energy content (or the internal energy) of the element. We can write it mathematically as

$$E_{in} - E_{out} + E_{gen} = E_{st} \quad ...(3.1)$$

where, E_{in} = energy entering the control volume per unit time

E_{out} = energy leaving the control volume per unit time

E_{gen} = energy generated within the control volume per unit time

E_{st} = energy storage within the control volume per unit time.

Let us calculate these quantities, one by one.

To calculate E_{in}. Energy enters the differential control volume from all the three sides by conduction only, since the control volume is embedded within the body considered.

Let the energy entering the control volume in the X-direction through face ABCD be Q_X. Similarly, Q_Y and Q_Z enter the control volume from the faces ABFE and DAEH as shown in the Fig. 3.1.

$$E_{in} = Q_x + Q_y + Q_z \quad ...(3.2)$$

To calculate E_{out}. Energy entering the control volume in the X-direction at face ABCD leaves the control volume at the opposite face EFGH. This is designated as Q_{x+dx}. Similarly, Q_{y+dy} and Q_{z+dz} leave the control volume from the surfaces opposite to the ones at which they entered. Therefore, we write,

$$E_{out} = Q_{x+dx} + Q_{y+dy} + Q_{z+dz} \quad ...(3.3)$$

Now, from calculus, we know that Q_{x+dx} etc. can be expressed by a Taylor series expansion, where, neglecting the higher order terms, we can write,

$$Q_{x+dx} = Q_x + \frac{\partial Q_x}{\partial x} \cdot dx \quad ...(3.4a)$$

$$Q_{y+dy} = Q_y + \frac{\partial Q_y}{\partial y} \cdot dy \quad ...(3.4b)$$

$$Q_{z+dz} = Q_z + \frac{\partial Q_z}{\partial z} \cdot dz \quad ...(3.4c)$$

To calculate E_{gen}. Let there be uniform heat generation within the volume at a rate of q_g (W/m³). Heat generation is a *volume phenomenon*, i.e. heat is generated throughout the bulk of the body—so, note its units (W/m³). As

GENERAL DIFFERENTIAL EQUATIONS FOR HEAT CONDUCTION

mentioned earlier, heat may be generated within the body due to passage of an electric current, a chemical reaction, nuclear fission, etc. Then, for the differential control volume $dx.dy.dz$, we can write,

$$E_{gen} = q_g dx\,dy\,dz \qquad ...(3.5)$$

To calculate E_{st}. As a result of the net energy flow into the control volume from all the three directions and the heat generated within the control volume itself, internal energy of the control volume increases. This will manifest itself as an increase in the temperature of the control volume. Let the temperature of the control volume increase by dT in time $d\tau$. Then, if ρ is the density and c_p, the specific heat of the material of the control volume, rate of increase of internal energy of control volume is given by,

$$E_{st} = \rho\,dx\,dy\,dz \cdot c_p \frac{\partial T}{\partial \tau} \qquad ...(3.6)$$

Now, substituting for all terms in Eq. 3.1, we get,

$$E_{in} - E_{out} + E_{gen} = E_{st}$$

i.e.

$$(Q_x + Q_y + Q_z) - (Q_{x+dx} + Q_{y+dy} + Q_{z+dz}) + q_g.dx.dy.dz = \rho\,dx\,dy\,dz \cdot c_p \frac{\partial T}{\partial \tau}$$

i.e.

$$-\left(\frac{\partial Q_x}{\partial x}\cdot dx + \frac{\partial Q_y}{\partial y}\cdot dy + \frac{\partial Q_z}{\partial z}\cdot dz \right) + q_g dx\,dy\,dz = \rho\,c_p \frac{\partial T}{\partial \tau} dx\,dy\,dz \qquad ...(3.7)$$

Now, let us bring in Fourier's law of heat conduction. If, for generality, we assume k_x, k_y, k_z to be the thermal conductivities of the material in the x, y and z-directions respectively, and A_x, A_y and A_z to be the areas normal to the respective heat flow directions, we can write for the heat flow rates,

$$Q_x = -k_x A_x \frac{\partial T}{\partial x} = -k_x dy\,dz \frac{\partial T}{\partial x} \qquad ...(3.8a)$$

$$Q_y = -k_y A_y \frac{\partial T}{\partial y} = -k_y dx\,dz \frac{\partial T}{\partial y} \qquad ...(3.8b)$$

$$Q_z = -k_z A_z \frac{\partial T}{\partial z} = -k_z dx\,dy \frac{\partial T}{\partial z} \qquad ...(3.8c)$$

Substituting Eq. (3.8) in (3.7), and dividing throughout by $dx.dy.dz$, we obtain,

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z} \right) + q_g = \rho\,c_p \frac{\partial T}{\partial \tau} \qquad ...(3.9)$$

This is the **general form of heat diffusion equation in Cartesian coordinates,** *for time dependent (i.e. unsteady state) heat conduction, with variable thermal conductivity and uniform heat generation within the body.* This is a very important basic equation for conduction analysis. It has to be solved with appropriate initial and boundary conditions to get the temperature distribution within the body as a function of spatial and time coordinates. Of course, the heat transfer rate is calculated applying the Fourier's law, once the temperature distribution is known.

Now, **if the material is isotropic,** i.e. the thermal conductivity is the same in all the three directions, i.e. $k_x = k_y = k_z = k$ say, then we can write,

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right) + q_g = \rho c_p \frac{\partial T}{\partial \tau} \qquad ...(3.10)$$

Eq. 3.10 is the **general form of heat diffusion equation in Cartesian coordinates,** *for time dependent (i.e. unsteady state) conduction, when thermal conductivity varies with temperature (i.e. with position) and uniform heat generation occurs within the body.*

If k is constant and does not vary with temperature, i.e. k does not change with position, the Eq. 3.10 can be written as,

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + q_g = \rho c_p \frac{\partial T}{\partial \tau}$$

i.e.
$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q_g}{k} = \frac{\rho c_p}{k}\frac{\partial T}{\partial \tau} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$ ...(3.11)

i.e.
$$\nabla^2 T + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$ ...(3.12)

where, $\alpha = k/(\rho c_p)$ is thermal diffusivity, and
∇ = Laplacian operator.

Solution of general form of heat diffusion equation as given in Eq. 3.10 or 3.12 is rather complicated. However, in many practical applications, we make simplifying assumptions and the resulting equations are easily solved. For example:

(i) **Steady state** This means that the temperature at any position does not change with time, i.e. $\frac{\partial T}{\partial \tau} = 0$. So, Eq. 3.12 becomes:

$$\nabla^2 T + \frac{q_g}{k} = 0$$ ...(3.13)

This is known as **Poisson equation** and is for steady state, three-dimensional heat conduction with heat generation, with constant thermal conductivity, in Cartesian coordinates.

(ii) **With no internal heat generation** This means that q_g term is zero. So, Eq. 3.12 becomes,

$$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$ ...(3.14)

This is known as **Diffusion equation**, and it represents time dependent, three-dimensional heat conduction, with no internal heat generation, and with constant thermal conductivity, in Cartesian coordinates.

(iii) **Steady state, with no internal heat generation** This means that q_g and $\frac{\partial T}{\partial \tau}$ are zero. So, Eq. 3.12 becomes,

$$\nabla^2 T = 0$$ ...(3.15)

This is known as **Laplace equation**, and it represents steady state, three-dimensional heat conduction with no internal heat generation, with constant thermal conductivity, in Cartesian coordinates.

(iv) **One-dimensional, steady state, with no internal heat generation** This means that,

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial y^2} = 0; \quad q_g = 0 \text{ and } \frac{\partial T}{\partial \tau} = 0$$

So, Eq. 3.12 becomes,

$$\frac{d^2 T}{dx^2} = 0.$$ ...(3.16)

Note that now, partial derivative is written as full derivative since temperature is dependent on one coordinate only.

You may be wondering why we have to consider one-dimensional heat flow when we are dealing with three-dimensional bodies. You will be surprised to know that solution of this simplified version of heat conduction equation for cases of simple geometries gives results with acceptable accuracy for engineering applications. One-dimensional conduction implies that temperature gradient is considerable only in one direction and is relatively negligible in the other three directions; consequently, heat flow is also in only one direction. Examples of such practical cases are: large slab (or wall) where length in one dimension (say, its thickness) is small compared to the other two-dimensions—then, temperature varies only along its thickness; long cylinder, whose temperature may be assumed to vary only along its radius; sphere, whose temperature may be assumed to vary only along its radius.

3.3 General Differential Equation for Heat Conduction in Cylindrical Coordinates

Eq. 3.10 derived earlier is suitable to analyse heat transfer in regular bodies of rectangular, square or parallelopiped shapes. But, if we have to analyse heat transfer in cylindrical-shaped bodies (which are commonly used in practice), then, working with cylindrical coordinates is more convenient, since in that case, the coordinate axes match with the system boundaries.

Nomenclature for cylindrical coordinate system is shown in Fig. 3.2.
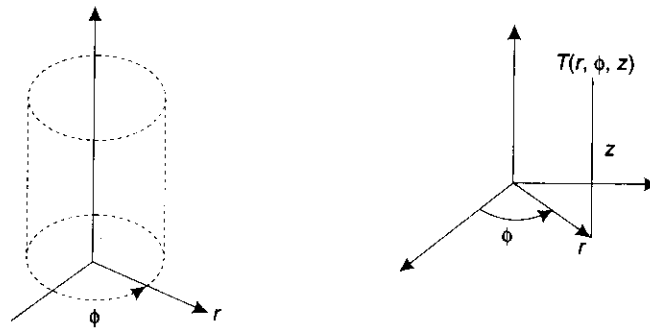
FIGURE 3.2 Nomenclature for derivation of general differential equation for heat conduction in cylindrical coordinate system

Differential equation for heat conduction in cylindrical coordinates may be derived by considering an elemental cylindrical control volume of thickness dr and making an energy balance over this control volume, as was done in the case of Cartesian coordinates, or, coordinates transformation can be adopted; for this purpose, transformation equations are,

$$x = r \cos\phi$$
$$y = r \sin\phi$$
$$z = z$$
$$\phi = \tan^{-1}(y/x)$$

The resulting general differential equation in cylindrical coordinates is,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad ...(3.17)$$

Eq. 3.17 is the **general differential equation in cylindrical coordinates,** *for time dependent, three-dimensional conduction, with constant thermal conductivity and with internal heat generation.*

For one-dimensional conduction in r direction only, we get from Eq. 3.17,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad ...(3.18)$$

i.e.

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad ...(3.19)$$

Eq. 3.19 represents **one-dimensional, time dependent conduction in r direction only, with constant k and uniform internal heat generation, in cylindrical coordinates.**

And, **for steady state, one-dimensional heat conduction in r direction only, with constant k and uniform heat generation,** Eq. 3.19 reduces to,

$$\frac{d^2 T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{q_g}{k} = 0. \qquad ...(3.20)$$

3.4 General Differential Equation for Heat Conduction in Spherical Coordinates

To analyse heat transfer in spherical systems, working with spherical coordinates is more convenient, since the coordinate axes match with system boundaries. Nomenclature for the spherical coordinates is shown in Fig. 3.3.
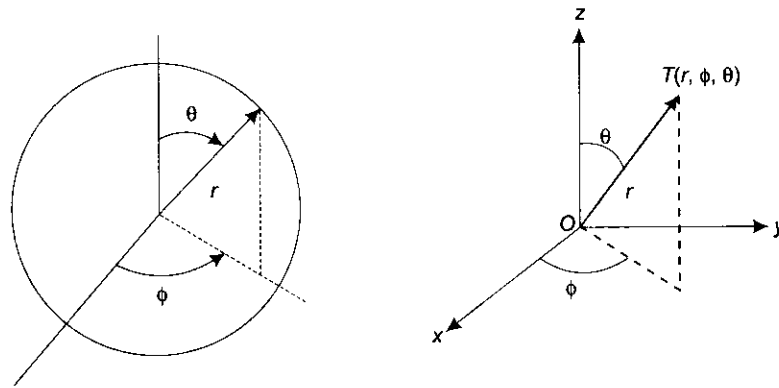

FIGURE 3.3 Nomenclature for derivation of general differential equation for heat conduction in spherical coordinate system

Differential equation for heat conduction in spherical coordinates may be derived by considering an elemental spherical control volume and making an energy balance over this control volume, as was done in the case of Cartesian and cylindrical coordinates, or, coordinate transformation can be adopted using the following transformation equations,

$$x = r \sin \theta \sin \phi$$
$$y = r \sin \theta \sin \phi$$
$$z = r \cos \theta$$

The resulting general differential equation in spherical coordinates is,

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 T}{\partial \phi^2} + \frac{q_g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial \tau} \qquad ...(3.21)$$

Eq. 3.21 is the **general differential equation in spherical coordinates,** *for time dependent, three-dimensional conduction, with constant thermal conductivity and with internal heat generation.*

For one-dimensional conduction in r direction only, we get from Eq. 3.21,

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{q_g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial \tau} \qquad ...(3.22)$$

i.e.

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{q_g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial \tau} \qquad ...(3.23)$$

Eq. 3.23 represents one-dimensional, time dependent conduction in r direction only, with constant k and uniform internal heat generation, in spherical coordinates.

And, for steady state, one-dimensional heat conduction in r direction only, with constant k and uniform heat generation, Eq. 3.23 reduces to,

$$\frac{d^2 T}{dr^2} + \frac{2}{r} \frac{dT}{dr} + \frac{q_g}{k} = 0. \qquad ...(3.24)$$

3.5 Boundary and Initial Conditions

Let us rewrite the heat diffusion eqn. i.e. Eq 3.11 in Cartesian coordinates,

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

Of course, temperature distribution in the body is obtained by solving this differential equation.

We observe that this is a second order differential equation in spatial coordinates; therefore, its solution will need two boundary conditions to eliminate the two constants of integration. Also, this equation is of first order in time coordinate; so, it will require one initial condition, i.e. the condition at $\tau = 0$.

Commonly encountered Boundary Conditions (B.C.'s) are:

    (i) Prescribed temperature conditions at the boundaries—known as **B.C. of the first kind** or Dirichlet condition

    (ii) Prescribed heat flux condition at the boundaries—known as **B.C. of the second kind** or Neumann condition

    (iii) Convection boundary condition—known as **B.C. of the third kind**

    (iv) Interface boundary condition—known as **B.C. of the fourth kind.**

We will study below the method of representing the boundary conditions mathematically.

3.5.1 Prescribed Temperatures at the Boundaries (B.C. of the First Kind)

This situation is represented in Fig. 3.4. Here, it is granted that the temperatures at the boundaries are specified and

FIGURE 3.4 Prescribed temperatures at the boundaries (B.C. of the first kind)

are constant. For example, temperature at a surface is constant if that surface is in contact with a melting solid or boiling liquid. Or, in more general case, the variation of temperature at the surface may be specified as a function of position and time. Referring to Fig. 3.4, the surface at $x = 0$ is maintained at a constant temperature T_1 and the surface at $x = L$ is maintained at constant temperature T_2.
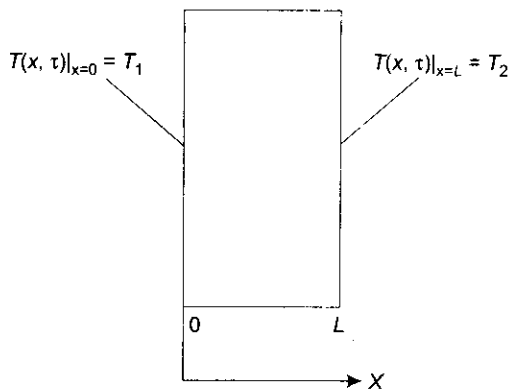
Mathematically, these conditions are represented as:

$$T(x, \tau)\,|_{x=0} = T(0, \tau) = T_1 \qquad ...(3.25\ a)$$
$$T(x, \tau)\,|_{x=L} = T(L, \tau) = T_2. \qquad ...(3.25\ b)$$

3.5.2 Prescribed Heat Flux at the Boundaries (B.C. of the Second Kind)

Here, the heat flux at the boundaries is assumed to be known. For example, if a surface is heated by an electric heater, the heat flux entering the surface is known. This condition is depicted in Fig. 3.5 (a).

In Fig. 3.5 (a), a plate of thickness L is shown. At $x = 0$, i.e. at the left face, a heat flux q_0 is supplied; this is conducted into the material as shown. At $x = L$, i.e. at the right face, a heat flux q_L is supplied and this is also conducted into the material. This situation is mathematically represented as follows, remembering that the conduction flux at a surface is equal to the heat flux supplied. Also, note clearly that $-k.(\partial T/\partial x)$ represents the heat flux in the positive X-direction, i.e. from left to right; if the direction of heat flux in a slab is from right to left, obviously, it is equal to $+k(\partial T/\partial x)$.

At $x = 0$: $\qquad\qquad q_0 = -k.(\partial T/\partial x)\,|_{x=0}$ $\qquad\qquad\qquad\qquad$ ...(3.26 a)

At $x = L$: $\qquad\qquad q_L = +k.(\partial T/\partial x)\,|_{x=L}$ $\qquad\qquad\qquad\qquad$ ...(3.26 b)

Note again that in Eq. 3.26 b, RHS is positive since the heat flux at $x = L$ is in the negative X-direction, i.e. from right to left as shown in the Fig. 3.5 (a). Similar relations can be written if the geometry is cylindrical or spherical.

There are **two special cases** of this boundary condition,

    (i) **Insulated boundary** Many times, to reduce the heat loss (or gain), the boundary is insulated with an appropriate insulating material. Even though theoretically heat loss will be reduced to zero only with an infinitely thick insulation thickness, heat loss may be practically assumed to be zero with a sufficiently thick insulation; we call this as *perfect insulation.*
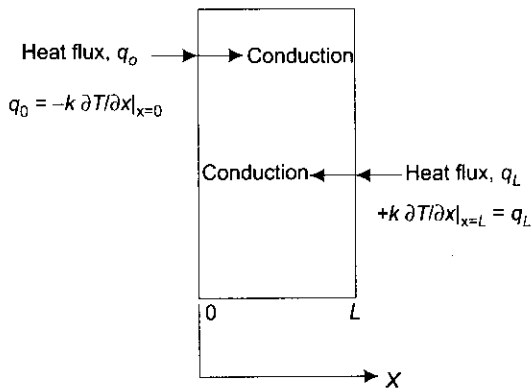
FIGURE 3.5(a) Prescribed heat flux at the boundaries (B.C. of second kind)
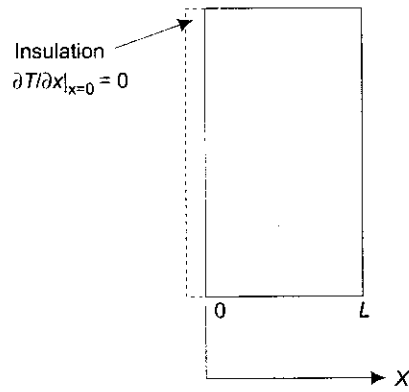
FIGURE 3.5(b) Insulated boundary at $x = 0$

So, for a perfectly insulated boundary at $x = 0$, shown in Fig. 3.5 (b), the heat flux across the boundary is zero and we represent this condition mathematically as follows,

$$k.(\partial T(0, \tau)/\partial x) = 0$$

or, $(\partial T(0, \tau)/\partial x) = 0$ ...(3.26c)

(ii) **Thermal symmetry** In many cases, there is thermal symmetry over a plane inside the system being analysed; for example, consider a copper plate, initially heated to a high temperature and then hung in air for cooling. It is intuitively clear that heat flow is from the centre of the plate to the two sides and the centre plane will be the plane of symmetry. In other words, no heat will cross this plane, i.e. this plane is equivalent to an insulated boundary.

So, for the centre plane, we can write,

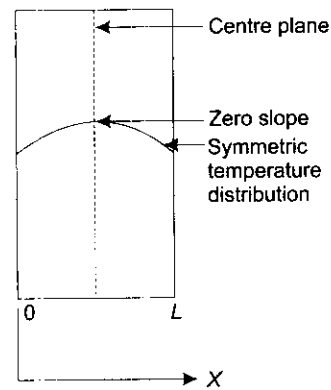$$[\partial T(L/2, \tau)/\partial x] = 0 \quad ...(3.26d)$$

FIGURE 3.5(c) Thermal symmetry at $x = L/2$

3.5.3 Convection Boundary Condition (B.C. of the Third Kind)

This is a more common practical situation, where heat transfer occurs at the boundary surface to or from a fluid flowing on the surface at a known temperature and a known heat transfer coefficient, e.g. in heat exchangers, condensers, reboilers etc.

Consider, again, a slab of thickness L as shown in Fig. 3.6.

At the left surface ($x = 0$), a hot fluid of temperature T_1 is flowing with a heat transfer coefficient h_1, supplying heat into the body. At the right surface ($x = L$), a cold fluid at a temperature T_2 is flowing on the surface, removing heat from the body with a heat transfer coefficient h_2.

Equating the conduction heat flux to the convection heat flux on either surface and remembering to note the direction of heat flow (i.e. whether it is in the positive X-direction or negative X-direction), we can represent this boundary condition mathematically as follows,

At $X = 0$: $h_1(T_1 - T\vert_{X=0}) = -k.(\partial T/\partial x)\vert_{x=0}$ ...(3.27 a)

At $X = L$: $h_2(T\vert_{X=L} - T_2) = -k.(\partial T/\partial x)\vert_{X=L}$ ...(3.27 b)

In Eq. 3.27 b, we write for the conduction heat flux at $x = L$: $-k.(\partial T/\partial x)\vert_{X=L}$ since heat is flowing in positive X-direction.

Using the same principles, expressions can be written for convection boundary conditions at surfaces of cylindrical and spherical geometries.
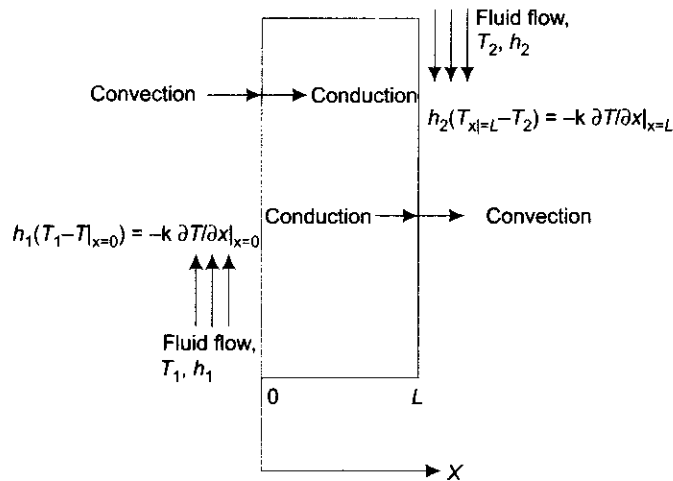
GENERAL DIFFERENTIAL EQUATIONS FOR HEAT CONDUCTION **33**

FIGURE 3.6 Convection boundary condition (B.C. of the third kind)

3.5.4 Interface Boundary Condition (B.C. of the Fourth Kind)

When a system is made up of one or more layers of different materials, solution of the problem requires that the conditions at the interface between the layers A and B is specified. *Perfect thermal contact* at the interface presupposes the following requirements:

FIGURE 3.7 Interface boundary condition (B.C. of the fourth kind)

(i) both the bodies must have the same temperature at the interface

(ii) heat flux on both the sides of the interface must be same.

Interface boundary condition is depicted in Fig. 3.7.

We write, at the interface,

$$T_A = T_B \qquad \qquad ...(3.28a)$$

$$-k_A.(\partial T / \partial x) = -k_B.(\partial T / \partial x) \qquad ...(3.28b)$$

Of course, the four boundary conditions explained above do not cover all the possible boundary conditions that may be encountered in practice. However, in any given situation, correct B.C. can be derived by applying the energy balance at the surface (i.e. to a control volume of zero volume—which means that no energy storage is possible at the control surface—and, heat entering IN = heat going OUT), as was done in deriving Eq. 3.27.
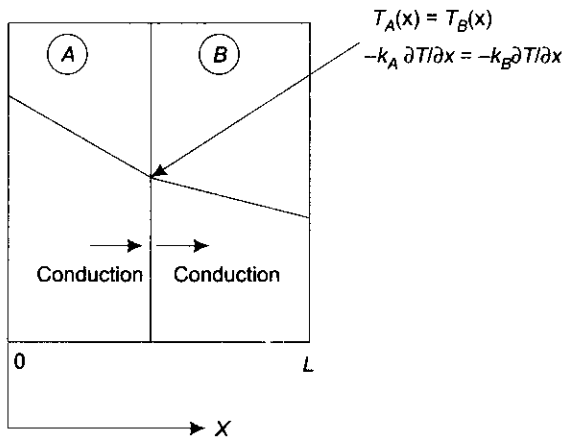
As a further example of this technique, consider a slab of thickness L; at its left surface, it receives heat by radiation and at its right face, loses heat by radiation. This situation is represented mathematically as shown in Fig. 3.8.

On the left hand side, energy impinging on the surface by radiation is equated to the energy conducted into the slab; since the heat is conducted in the positive X-direction the conduction term (flux) has a negative sign as per Fourier's law. Similarly, on the right hand face, radiation impinging on the surface is conducted into the slab from right to left, i.e. in the negative X-direction; therefore, we put a positive sign in the conduction term, as shown in the Fig. 3.8.
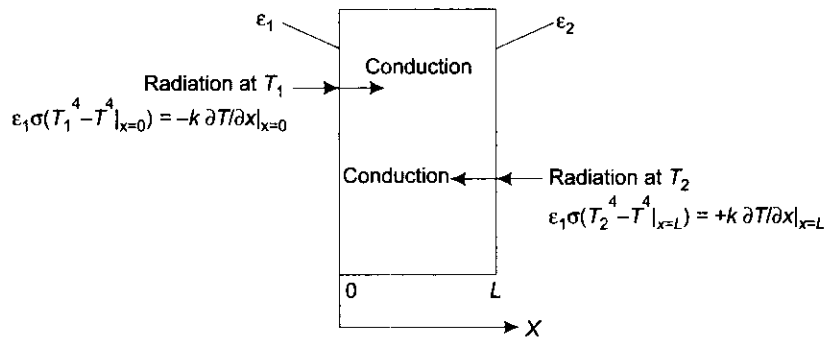
FIGURE 3.8 Radiation boundary conditions at the surfaces

Example 3.1. Temperature variation in a slab is given by: $T(x) = 100 + 200\,x - 500\,x^2$, where x is in metres; $x = 0$ at the left face and $x = 0.3$ m at the right face. Thermal conductivity of the material $k = 45$ W/(mC). Also, $c_p = 4$ kJ/(kgK) and $\rho = 1600$ kg/m^3. Determine:

(i) Temperature at both surfaces
(ii) Heat transfer at left face and its direction
(iii) Heat transfer at right face and its direction
(iv) Is there any heat generation in the slab? If so, how much?
(v) Maximum temperature in the slab and its location
(vi) Time rate of change of temperature at X = 0.1 m if the heat generation rate is suddenly doubled
(vii) Draw the temperature profile in the slab
(viii) Average temperature of the slab.

Solution. Temperature profile is given; so, temperatures at the left and right faces are easily determined by substituting $x = 0$ and $x = 0.3$ m. Maximum temperature is determined by first differentiating $T(x)$ w.r.t. x and equating to zero to get the position (x max) where the maximum occurs and then substituting this x max in $T(x)$. Temperature profile is graphed using Mathcad. Time rate of change of temperature at X = 0.1 m is found by applying the time dependent, one dimensional, heat conduction equation in Cartesian coordinates. Procedure to determine the average temperature of the slab is explained at the end. We shall solve this problem in Mathcad, with suitable comments at each step.
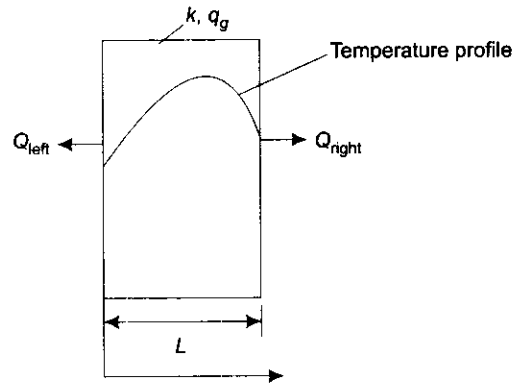
Data:

$$L := 0.3 \text{ m} \qquad k := 45 \text{ W/mC} \qquad c_p := 4000 \text{ J/kg K}$$

$$\rho := 1600 \text{ kg/m}^3 \qquad \alpha := \frac{k}{\rho \cdot c_p}$$

i.e. $$\alpha = 7.031 \times 10^{-6} \text{ m}^2/\text{s}$$

$$T(x) := 100 + 200x - 500x^2$$

FIGURE Example 3.1(a)

(Define $T(x)$... i.e. temperature as a function of x)

Temperature at left face, i.e. at $x = 0$ $T(0) = 100°$C
Temperature at right face, i.e. at $x = 0.3$ m: $T(0.3) = 115°$C

To find max. temperature

Define the first derivative of $T(x)$: $T'(x): = \dfrac{d}{dx} T(x)$

Also, define the second derivative of $T(x)$: $T''(x): = \dfrac{d}{dx} T'(x)$

By hand calculation: we get: $T'(x) = 200 - 1000.x$

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We set $T'(x)$ equal to zero to get the position x max where temperature is maximum

i.e. $200 - 1000.x = 0$.

This gives $x = 0.2$ m. Substitute this value of x max in $T(x)$ to get the value of $Tmax$.

So, $Tmax = T(0.2) = 100 + 200 \times 0.2 - 500 \times (0.2)^2 = 120°C$.

However, in Mathcad, all this procedure is very simple. Read the comments in Mathcad solution below.

Set $T'(x) = 0$ and find out the value of x max. To do this, use the root function, which solves the root of $T'(x) = 0$. First, assume a trial value of x; then use the root function which gives the true value of x

$$x := 0.15 \qquad \qquad (Trial\ value\ of\ x)$$

$$x\ \mathrm{max} := \mathrm{root}(T'(x),\ x) \qquad x\ max\ is\ obtained\ from\ the\ root\ function)$$

i.e. $\qquad x\ \mathrm{max} = 0.2$ m $\qquad \qquad (value\ of\ x\ where\ T\ is\ a\ max.)$

To get Tmax: Substitute this value of x max in $T(x)$:

$$T(x\ \mathrm{max}) = 120°C$$

To Sketch the temperature distribution in the slab:

$$x := 0, 0.01, ..., 0.3 \qquad (Define\ the\ range\ variable\ x,\ i.e.\ x\ to\ vary\ from\ 0$$
$$to\ 0.3\ m\ in\ steps\ of\ 0.01\ m)$$

To draw the graph:

Just select the $x - y$ plot from pallete, plug in x and $T(x)$ in the place holders:

x is in metres and $T(x)$ in deg. C. Click anywhere outside the graph region; immediately the graph appears.

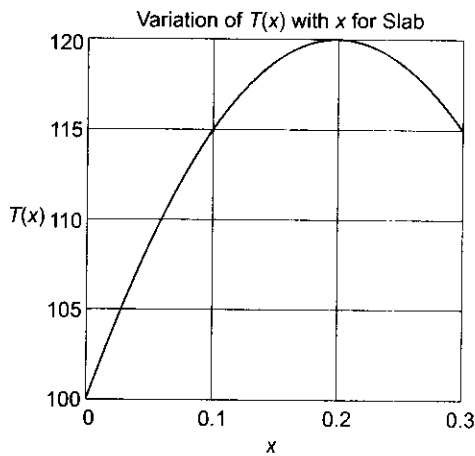Note from the graph that the maximum temperature occurs at $x = 0.2$ m and its value is 120°C, as already calculated.

FIGURE Example 3.1(b)

To calculate the heat fluxes at the left and right faces:

Apply the Fourier's law at $x = 0$ and $x = 0.3$ m, remembering that temperature gradient is given in $T'(x)$, already defined.

$$q_{\mathrm{left}} := -k \cdot T'(0) \qquad (applying\ Fourier's\ law$$
$$at\ left\ face,\ i.e.\ at\ x = 0)$$

i.e. $\qquad q_{\mathrm{left}} = -9 \times 10^3 \qquad$ **(Heat flux at the left face (W/m²);** note that $-ve$ sign indicates heat flowing from right to left)

$$q_{\mathrm{right}} := -k \cdot T'(0.3) \qquad (applying\ Fourier's\ law$$
$$at\ right\ face,\ i.e.\ at\ x = 0)$$

i.e. $\qquad q_{\mathrm{right}} = 4.5 \times 10^3 \qquad$ **(Heat flux at the right face (W/m²);** note that $+ve$ sign indicates heat flowing from left to right)

$$q_{\mathrm{total}} := |q_{\mathrm{left}}| + |q_{\mathrm{right}}| \qquad (Total\ heat\ generated\ per$$
$$m^2\ of\ surface)$$

i.e. $\qquad q_{\mathrm{total}} = 1.35 \times 10^4\ \mathrm{W/m^2} (Total\ heat\ generated/m^2)$

Therefore, q_g, the volumetric heat generated rate is given by total heat generated per unit volume:

$$q_g = \frac{q_{\mathrm{total}}}{1 \cdot 0.3}; \quad \text{i.e. } q_g = 4.5 \times 10^4\ \mathrm{W/m^3} \qquad (volumetric\ heat\ generation\ rate\ in\ the\ slab)$$

To calculate the time rate of change of temperature at $x = 0.1$ m when q_g is suddenly doubled:

We have the time dependent differential equation for heat conduction in Cartesian coordinates

$$\frac{\partial^2 T}{\partial x^2} + \frac{q_g}{k} = \frac{\rho\ c_p}{k} \frac{\partial T}{\partial \tau} \qquad \frac{\partial T}{\partial \tau} = \frac{1}{\alpha} \frac{\partial T}{\partial \tau}$$

Therefore,

$$\frac{\partial T}{\partial \tau} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{q_s}{k} \alpha$$

From the given equation for temperature distribution, it is clear that $\frac{\partial^2 T}{\partial x^2}$ does not depend on x, i.e. $\frac{\partial T}{\partial \tau}$ depends only on q_g:

$$\frac{dT}{d\tau(x)} := \alpha \cdot T''(x) + \alpha \cdot \frac{2 \cdot q_g}{k}$$

(define dT/dτ as a function x. Now, we can get dT/dτ at any x by simply substituting that value of x in the function defined)

i.e. $\dfrac{dT}{d\tau(0.1)} = 7.031 \times 10^{-3} \text{ C/s}$

(time rate of change of temp.)

Note that this is true for all x since T"(x) does not depend on x for the temperature distribution given

To determine the average temperature of the slab:

For a differential element of thickness dx, amount of heat energy contained in the element is equal to $A.dx.\rho.c_p.T(x)$. Total amount of energy in the slab is obtained by integrating this from $x = 0$ to $x = 0.3$. Now, if the average temperature of slab is T_{av}, amount of energy in the slab can also be written as: $\rho.A.L.c_p.T_{av}$. Equating these two expressions, we get

$$\rho A L c_p T_{av} = \int_0^L (A dx) \, \rho \, c_p T(x)$$

i.e.

$$T_{av} = \frac{1}{L}\int_0^L T(x)dx = \frac{1}{0.3}\int_0^{0.3} (100 + 200x - 500x^2)dx$$

i.e.

$$T_{av} = \frac{1}{0.3}\left[30 + 200 \times \frac{(0.3)^2}{2} - \frac{500}{3}(0.3)^3\right] = \frac{1}{0.3}[39 - 4.5] = 115°C$$

In Mathcad, evaluating the integral within given limits is very easy. First, define T_{avg} and then just plug in the limits; Mathcad automatically evaluates the integral and gives the value.

$$T_{avg} := \frac{1}{L} \cdot \int_0^L T(x)dx$$

(Mathcad easily does the integration of T(x) within the limits specified)

i.e. $T_{avg} = 115°C$

(Average temperature of the slab.)

Note that Mathcad directly gives the value of the integral within the limits specified; there is no need to expand the integral and write down as you do in hand calculations.

Example 3.2. Uniform internal heat generation at $q_g = 5 \times 10^7 \text{ W/m}^3$ occurs in a cylindrical nuclear reactor fuel rod of 50 mm diameter, and under steady state conditions the temperature distribution is of the form:

$T(r) = 800 - 4.167 \times 10^5 r^2$, where T is in deg. Celsius and r is in metres. The fuel rod properties are: $k = 30 \text{ W/(mK)}$, $\rho = 1100 \text{ kg/m}^3$ and $c_p = 800 \text{ J/(kgK)}$

(a) What is the rate of heat transfer per unit length of the rod at $r = 0$ (i.e. at the centre line) and at $r = 25$ mm (i.e. at the surface)?

(b) Sketch the temperature distribution along the radius.

(c) If the reactor power is suddenly increased to 10^8 W/m^3, what is the initial time rate of temperature change at $r = 0$ and $r = 25$ mm?

(d) Find the average temperature of the rod in the first case.

Solution. Here, temperature distribution is given; so, heat flux can be calculated at any radius r from Fourier's law: $q = -k$ (dT/dr). Temperature distribution along the radius is easily graphed with Mathcad. Time rate of change of temperature when q_g changes is found out by applying the time dependent, one-dimensional heat conduction equation in cylindrical coordinates. Average temperature of the cylinder is obtained from first principles as done in the case of slab in Example 3.1.

Data:

$R := 0.025 \text{ m}$

$k := 30 \text{ W/mC}$

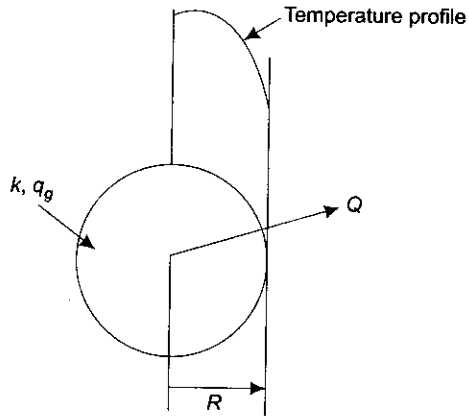$c_p := 800 \text{ J/kgK}$

$\rho := 1100 \text{ kg/m}^3$

$L := 1 \text{ m}$

FIGURE Example 3.2(a)

$T(r) := 800 - 4.167 \times 10^5 \cdot r^2$ *(Define T(r)... i.e. temperature as a function of r)*

$T'(r) := \dfrac{d}{dr} T(r)$ *(Define first derivative of T(r))*

$T''(r) := \dfrac{d}{dr} T'(r)$ *(Define second derivative of T(r))*

$Q_{centre} = 0$ **(heat transfer rate at the centre is zero since temperature at centre is maximum and dT/dr = 0 at r = 0.)**

To find the heat transfer rate at the surface (i.e. at r = 0.025 m):

Apply Fourier's law: $Q_{surface} = -k\,A_s\,(dT/dr)|_{at\,r=R}$

$T'(R) = -2.083 \times 10^4\ C/m$

(dT/dr at the surface, i.e. at r = R)

$Q_{surface} := k \cdot (2 \cdot \pi \cdot R \cdot L) \cdot T'(R)$

(heat transfer rate at the surface is obtained by applying Fourier's law at the surface, i.e. at r = R; T'(R) is the temperature gradient at r = R)

i.e. $Q_{surface} = 9.818 \times 10^4\ \textbf{W/metre length}$ *(heat transfer at the surface)*

Temperature distribution:

$r := 0, 0.001, ..., 0.025$ *(define a range variable, i.e. r varies from 0 to 0.025 m in steps of 0.001 m)*

Then, select the x–y graph from pallete and fill in the place holders in both the axes. On x-axis, fill in r and on y-axis, fill in T(r). Click anywhere outside the graph region and immediately, the graph appears.

To calculate the time rate of change of temperature at x = 0.1 m:

We have the one-dimensional, time dependent differential equation, with constant k, for heat conduction in cylindrical coordinates:

$$\frac{d^2 T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{q_s}{k} = \frac{1}{\alpha}\frac{dT}{d\tau}$$

Therefore, time rate of change of temperature is given by:

$$\frac{\partial T}{\partial \tau} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{q_s}{k}\right)$$
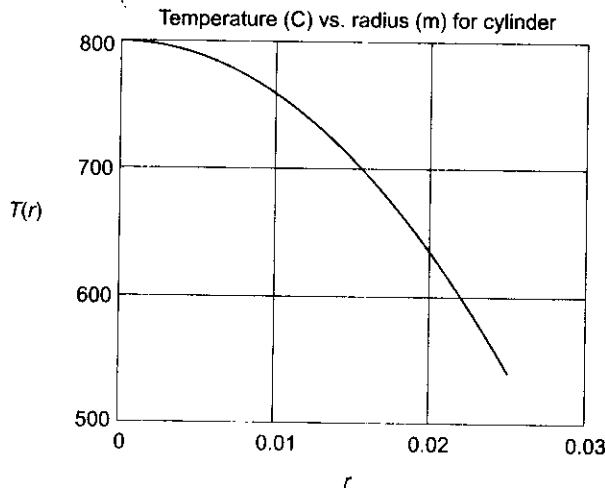

FIGURE Example 3.2(b)

Our aim is to find out $dT/d\tau$ when q_g changes suddenly to 10^8 W/m^3

$$q_g := 10^8 \text{ W/m}^3$$

$$\alpha := \frac{k}{\rho \cdot c_p} \quad \text{i.e.} \quad \alpha = 3.409 \times 10^{-5} \text{ m}^2/\text{s} \qquad \text{(thermal diffusivity.)}$$

$$T''(r) := \frac{d}{dr} T'(r) \qquad \text{...Define second derivative of } T(r) \text{ w.r.t. } r = d^2T/dr^2$$

$$\frac{dT}{d\tau(r)} := \alpha \cdot \left(T''(r) + \frac{1}{r} \cdot T'(r) + \frac{q_g}{k} \right) \qquad \left(\text{define } \frac{\partial T}{\partial \tau} \text{ the desired time rate of change of temperatire as a function of } r\right)$$

At the surface i.e. at $r = R$:

$$dT \text{ by } d\tau(r) := \alpha \cdot \left(T''(r) + \frac{1}{r} \cdot T'(r) + \frac{q_g}{k} \right) \qquad \left(\text{define } \frac{\partial T}{\partial \tau}, \text{ the desired time rate of change}\right)$$

i.e. $\qquad dT \text{ by } d\tau(0.025) = 56.814 \text{ C/s}$

At the centre, i.e. at $r = 0$:

$$dT \text{ by } d\tau(r) := \alpha \cdot \left(T''(r) + \frac{q_g}{k} \right) \qquad \text{(since at } r = 0, dt/dr = 0)$$

i.e. $\qquad dT \text{ by } d\tau(0) = 85.225 \text{ C/s.}$

To determine the average temperature of the cylinder:

For a differential element of thickness dr, amount of heat energy contained in the element is equal to $2\pi r.dr.L.\rho.c_p.T(r)$. Total amount of energy in the cylinder is obtained by integrating this from $r = 0$ to $r = R = 0.025$ m. Now, if the average temperature of cylinder is T_{av}, amount of energy in the cylinder can also be written as: $\rho.\pi.R^2.L.c_p.T_{av}$. Equating these two expressions, we get,

$$\pi R^2 L\rho c_p T_{av} = \int_0^R (2\pi r dr) L\rho c_p T(r)$$

i.e. $$T_{av} = \frac{2}{R^2} \int_0^R T(r) r dr$$

i.e. $$T_{av} = \frac{2}{R^2} \int_0^{0.025} (800 - 4.167 \times 10^5 r^2) r dr$$

i.e. $$T_{av} = \frac{2}{(0.025)^2} \times \left[800 \times \frac{(0.025)^2}{2} - 4.167 \times 10^5 \times \frac{(0.025)^4}{4} \right]$$

i.e. $$T_{av} = 669.78 °C.$$

All the above calculations are done just in one step easily in Mathcad:

$$T_{avg} := \frac{2}{R^2} \cdot \int_0^R T(r) r dr \qquad \left(\text{define } T_{avg} \text{ Mathcad easily does the integration of } T(r) \text{ within the limits specified}\right)$$

i.e. $\qquad T_{avg} = 669.781 °C \qquad \qquad$ *(Average temperature of the cylinder)*

Note that Mathcad directly gives the value of the integral within the limits specified; there is no need to expand the integral and write down as you do in hand calculations.

Example 3.3. Consider an orange, assumed to be a sphere of 8 cm diameter, producing an average internal heat generation of 2.25×10^4 W/m^3 during its ripening. Thermal conductivity of the material is 0.15 W/(mK) and its centre temperature is observed to be 50°C. Assuming one-dimensional, steady state conduction, find out:

    (i) temperature distribution along the radius,

    (ii) surface temperature,

    (iii) heat transferred at the surface of the sphere,

    (iv) draw the temperature profile along the radius, and

    (v) average temperature of the sphere.

Solution.

Data:

    $R := 0.04$ m $k := 0.15$ W/mC $q_g := 2.25 \times 10^4$ W/m^3

    $T_c := 50$ C...centre temperature

(i) Temperature distribution For steady state, one-dimensional conduction, for a sphere, we have the controlling differential equation:

FIGURE Example 3.3

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} + \frac{q_g}{k} = 0 \qquad ...(a)$$

To get the temperature distribution, we have to solve Eq. (a) with the following Boundary Conditions (B.C.'s):

    (i) at $r = 0$, $(dT/dr) = 0$, since the temperature has to be maximum at the centre because the heat flows from centre to periphery and symmetry considerations.

    (ii) Given: $T_c = 50$°C at the centre, i.e. at $r = 0$

Multiplying Eq. (a) by r^2:

$$r^2\frac{d^2T}{dr^2} + 2r\frac{dT}{dr} + \frac{q_g r^2}{k} = 0$$

i.e.

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{q_g r^2}{k} = 0$$

Integrating, we get,

or,

$$r^2\frac{dT}{dr} + \frac{q_g r^3}{3k} = C_1$$

or,

$$\frac{dT}{dr} = -\frac{q_g r}{3k} + \frac{C_1}{r^2} \qquad ...(b)$$

Integrating again,

$$T(r) = \frac{-q_g r^2}{6k} - \frac{C_1}{r} + C_2 \qquad ...(c)$$

Applying B.C. (i) to Eq. b: $C_1 = 0$

Then,

$$T(r) = \frac{-q_g r^2}{6k} + C_2$$

Applying B.C. (ii) to Eq. c: $C_2 = 50$

Substituting values of C_1 and C_2 in Eq. c, we get the temperature distribution in the sphere:

$$T(r) = \frac{-q_g r^2}{6k} + 50. \qquad ...(d)$$

(ii) Surface temperature Now, temperature at the surface is obtained simply by putting $r = R = 0.04$ m in Eq. d. It is easier to work in Mathcad; first, define the function $T(r)$:

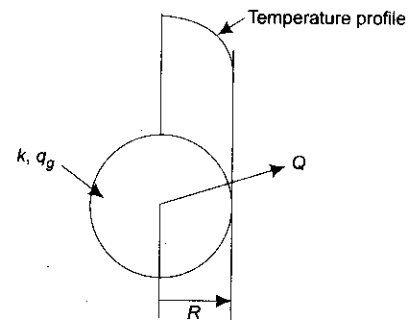$$T(r) := \frac{-q_g r^2}{6k} + 50 \qquad\qquad (Define\ T(r)...i.e.\ temperature\ as\ a\ function\ of\ r)$$

Surface temperature:

$$T(R) = 10°C \qquad \qquad \text{(surface temperature, i.e. at } r = 0.04 \ m\text{)}$$

(iii) Next, heat transfer at the surface This is determined by Fourier's law since we already have the relation for the temperature distribution:

i.e. $Q(r) = -k.(4\pi R^2).(dT/dr)|_{r=R}$. Gradients such as dT/dr and d^2T/dr^2 etc. are easily found in Mathcad, once the function $T(r)$ is defined:

Heat transfer at the surface:

$$T'(r) := \frac{d}{dr}T(r) \qquad\qquad \text{(Define derivative of } T(r)\text{)}$$

$$Q(r) := k\cdot(4 \cdot \pi \cdot R^2)\cdot T'(r) \qquad\qquad (Q(r) \text{ is the heat transfer rate at radius } r, \text{ by Fourier's law)}$$

i.e. $\qquad Q(R) = 6.032 \ W \qquad\qquad$ **(heat transfer rate at the surface, i.e. at $r = R = 0.04 \ m$)**

Check: this must be equal to heat generated inside the orange in steady state Q_{gen}.

$$Q_{gen} = q_g \cdot \frac{4}{3} \cdot \pi \cdot R^3$$

i.e. $\qquad Q_{gen} = 6.032 \ W...\text{checks.}$

Sketch the temperature profile along the radius This is done very easily and conveniently in Mathcad. First, define a range variable r from 0 to $R = 0.04$ m, in steps of say, 0.001 m; then, select the $x - y$ graph from the pallete and just fill in the place holders, i.e. fill in r in the place holder on the x-axis and $T(r)$ in the place holder on the y-axis. Click anywhere outside the graph region and immediately, the graph appears:

Temperature profile along the radius:

$$r := 0, 0.001.. 0.04 \qquad\qquad \text{(define the range variable, i.e. } r \text{ to vary from 0 to 0.04 } m,$$
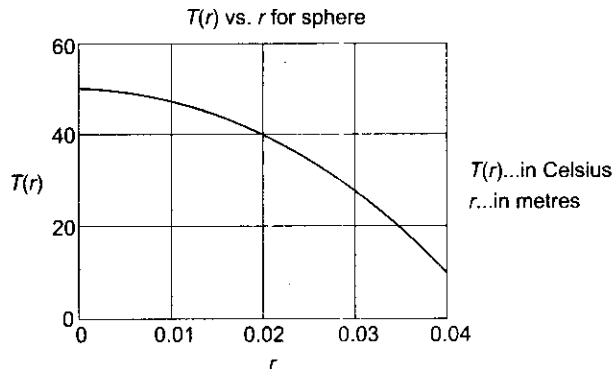$$\text{in steps of 0.001 } m\text{)}$$

FIGURE Example 3.3(b)

Note from the graph that maximum temperature occurs at the centre ($r = 0$); slope of the temperature curve, (dT/dr) tends to zero (i.e. the curve becomes almost horizontal) as it aproaches the y-axis at $r = 0$.

(v) Average temperature of the sphere For a differential element of thickness dr, amount of heat energy contained in the element is equal to $4\pi r^2.dr.\rho.c_p.T(r)$. Total amount of energy in the sphere is obtained by integrating this from $r = 0$ to $r = R$. Now, if the average temperature of sphere is T_{av}, amount of energy in the sphere can also be written as: $\rho.(4/3)\pi.R^3.c_p.T_{av}$. Equating these two expressions we get,

$$\frac{4}{3}\pi R^3 \rho c_p T_{av} = \int_0^R (4\pi r^2 \, dr)\rho c_p \, T(r)$$

i.e.

$$T_{av} = \frac{3}{R^3}\int_0^R T(r)r^2 \, dr$$

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i.e.
$$T_{av} = \frac{3}{(0.04)^3} \int_0^{0.04} \left(-q_g \frac{r^2}{6k} + 50 \right) r^2 \, dr$$

i.e.
$$T_{av} = \frac{3}{(0.04)^3} \left[50 \times \frac{(0.04)^3}{3} - 2.25 \times 10^4 \times \frac{(0.04)^5}{5 \times 6 \times 0.15} \right]$$

i.e.
$$T_{av} = 26°C.$$

All the above calculations are done just in one step easily in Mathcad:

$$T_{avg} := \frac{3}{R^3} \cdot \int_0^R T(r) r^2 \, dr$$

(define T_{avg} Mathcad easily does the integration of $T(r)$ within the limits specified)

i.e. $\qquad T_{avg} = 26°C$ $\hspace{3cm}$ (**Average temperature of the sphere**)

Note that Mathcad directly gives the value of the integral within the limits specified; there is no need to expand the integral and write down as you do in hand calculations.

3.6 Summary of Basic Equations

TABLE 3.1

Sl. No.	Equation	Remarks
1	$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) + q_g = \rho c_p \frac{\partial T}{\partial \tau}$$	Three-dimensional, time dependent heat conduction equation with heat generation and temperature dependent k, in Cartesian coordinates.
2	$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q_g}{k} = \frac{\rho c_p}{k}\frac{\partial T}{\partial \tau} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$	Three-dimensional, time dependent heat conduction equation with heat generation and constant k, in Cartesian coordinates.
3	$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q_g}{k} = 0$$	**Poisson equation, i.e.** three-dimensional, steady state heat conduction equation with heat generation and constant k, in Cartesian coordinates.
4	$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$	**Diffusion equation, i.e.** three-dimensional, time dependent heat conduction equation with no heat generation and constant k, in Cartesian coordinates.
5	$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$	**Laplace equation,** i.e. three-dimensional, steady state heat conduction equation. with no heat generation and with constant k, in Cartesian coordinates.
6	$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$	Three-dimensional, time dependent heat conduction equation with heat generation and constant. k, in cylindrical coordinates.
7	$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \\ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$	Three-dimensional, time dependent heat conduction equation with heat generation and constant k, in spherical coordinates.

Contd.

8	$\dfrac{\partial}{\partial x}\left(k\dfrac{\partial T}{\partial x}\right)+q_g=\rho c_p\dfrac{\partial T}{\partial \tau}$	One-dimensional, time dependent heat conduction equation with heat generation and temperature dependent k, in Cartesian coordinates.
9	$\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(rk\dfrac{\partial T}{\partial r}\right)+q_g=\rho c_p\dfrac{\partial T}{\partial \tau}$	One-dimensional, time dependent heat conduction equation with heat generation and temperature dependent k, in cylindrical coordinates.
10	$\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2k\dfrac{\partial T}{\partial r}\right)+q_g=\rho c_p\dfrac{\partial T}{\partial \tau}$	One-dimensional, time dependent heat conduction equation with heat generation and temperature dependent k, in spherical coordinates.
11	Equations 8, 9, 10 are compactly written as, $\dfrac{1}{r^n}\dfrac{\partial}{\partial r}\left(r^n k\dfrac{\partial T}{\partial r}\right)+q_g=\rho c_p\dfrac{\partial T}{\partial \tau}$ where, $n=0$ for Cartesian coordinates, use x as variable instead of r $n=1$ for cylindrical coordinates. $n=2$ for spherical coordinates.	Compact form of one-dimensional, **time dependent**, heat conduction equation with heat generation and **temperature dependent k**
12	$\dfrac{1}{r^n}\dfrac{\partial}{\partial r}\left(r^n\dfrac{\partial T}{\partial r}\right)+\dfrac{q_g}{k}=\dfrac{1}{\alpha}\dfrac{\partial T}{\partial \tau}$ where, $n=0$ for Cartesian coordinates, use x as variable instead of r $n=1$ for cylindrical coordinates. $n=2$ for spherical coordinates.	Compact form of one-dimensional, **time dependent**, heat conduction equation with heat generation and **constant k**
13	$\dfrac{1}{r^n}\dfrac{\partial}{\partial r}\left(r^n\dfrac{\partial T}{\partial r}\right)+\dfrac{q_g}{k}=0$ where, $n=0$ for Cartesian coordinates, use x as variable instead of r $n=1$ for cylindrical coordinates. $n=2$ for spherical coordinates.	Compact form of one-dimensional, **steady state**, heat conduction equation with heat generation and **constant k**
14	$\dfrac{d^2T}{dx^2}+\dfrac{q_g}{k}=0$	Alternate form of one-dimensional, **steady state**, heat conduction equation with heat generation and **constant k, in Cartesian coordinates.**
15	$\dfrac{d^2T}{dr^2}+\dfrac{1}{r}\left(\dfrac{dT}{dr}\right)+\dfrac{q_g}{k}=0$	Alternate form of one-dimensional, **steady state**, heat conduction equation with heat generation and **constant k, in cylindrical coordinates.**
16	$\dfrac{d^2T}{dr^2}+\dfrac{2}{r}\left(\dfrac{dT}{dr}\right)+\dfrac{q_g}{k}=0$	Alternate form of one-dimensional, **steady state**, heat conduction equation with heat generation and **constant k, in spherical coordinates.**
17	$\dfrac{d^2T}{dx^2}+\dfrac{q_g}{k}=\dfrac{1}{\alpha}\left(\dfrac{dT}{d\tau}\right)$	Alternate form of one-dimensional, time dependent, heat conduction equation with heat generation and **constant k, in Cartesian coordinates.**

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18	$\dfrac{d^2T}{dr^2} + \dfrac{1}{r}\left(\dfrac{dT}{dr}\right) + \dfrac{q_g}{k} = \dfrac{1}{\alpha}\left(\dfrac{dT}{d\tau}\right)$	Alternate form of one-dimensional, time depend-ent, heat conduction equation with heat genera-tion and **constant** k, **in cylindrical coordinates.**
19	$\dfrac{d^2T}{dr^2} + \dfrac{2}{r}\left(\dfrac{dT}{dr}\right) + \dfrac{q_g}{k} = \dfrac{1}{\alpha}\left(\dfrac{dT}{d\tau}\right)$	Alternate form of one-dimensional, time depend-ent, heat conduction equation with heat genera-tion and **constant** k, **in spherical coordinates.**

3.7 Summary

This chapter lays the foundation for the study of heat transfer by conduction. First, general differential equation for conduction was derived in Cartesian (or, rectangular) coordinates. This equation has to be solved for a given sys-tem applying the appropriate boundary and initial conditions to get the temperature field. To do this, mathematical representation of more common types of boundary and initial conditions are explained. Once the temperature distribution within the body is known, rate of heat transfer (or heat flux) at any point is calculated easily by apply-ing Fourier's law. Cartesian coordinates are used while dealing with rectangular geometries such as squares, rec-tangles, walls, parallelopipes, etc; these geometries find applications in furnaces, boiler walls, walls of buildings, air conditioning ducts, etc. Next, general differential equations for conduction in cylindrical and spherical systems are stated. They are useful in solving heat transfer problems involving cylindrical tanks, pipes, spherical storage ves-sels, reactors, etc. Summary of the basic relations is given in Tabular form for ready reference.

In engineering practice, we ordinarily deal with three-dimensional objects; however, solution of three-dimen-sional general differential equation is rather complicated. So, a simplifying assumption is made sometimes, of one-dimensional conduction, i.e. temperature variation is substantial only in one-dimension and the temperature variation is considered to be negligible in the other two-dimensions. Many practical problems fit into this category: e.g. walls whose thicknesses are small compared to other dimensions, long cylinders, spheres, etc. In such cases, analytical solutions for one-dimensional heat transfer problems are very much simplified.

In the next chapter, we shall study one-dimensional, steady state conduction as applied to a few regular geometries such as slabs, cylinders and spheres.

Questions

1. Derive the general differential equation in rectangular coordinates (i.e. Cartesian coordinates). Therefrom, write down the governing differential equations for the following cases:
   (i) 3-dimensional, constant k, unsteady state conduction with heat generation
   (ii) 3-dimensional, constant k, steady state conduction without heat generation
   (iii) 3-dimensional, temperature dependent k, steady state conduction with heat generation
   (iv) One-dimensional, constant k, unsteady state conduction with heat generation
   (v) One-dimensional, temperature dependent k, unsteady state conduction with heat generation
   (vi) One-dimensional, constant k, steady state conduction without heat generation
   (vii) One-dimensional, constant k, steady state conduction with heat generation.
2. Derive the general equation for the 3-dimensional unsteady state heat conduction with uniform rate of heat gen-eration in an isotropic solid. Hence, deduce Laplace's equation.

[V.T.U., Aug. 2001]

3. Write down the two-dimensional, steady state heat conduction equation in x and y variables in rectangular coor-dinate system, for the case of temperature dependent k and with uniform heat generation in the body.
4. Write down the one-dimensional, time dependent heat conduction equation in spherical and cylindrical coordinate systems, in the r variable, with temperature dependent k and with uniform heat generation in the body.
5. In a medium, heat conduction equation is given in the following form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + q_s = 0$$

   (a) Is the heat transfer steady or transient?
   (b) Is heat transfer one, two or three-dimensional?

(c) Is there heat generation in the medium?

(d) Is the thermal conductivity of the medium constant or variable with temperature?

6. In a medium, heat conduction equation is given in the following form:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

(a) Is the heat transfer steady or transient?

(b) Is heat transfer one, two or three-dimensional?

(c) Is there heat generation in the medium?

(d) Is the thermal conductivity of the medium constant or variable with temperature?

7. Explain what do you understand by 'one-dimensional heat conduction'.

8. State the general differential equation for steady state heat conduction in cylindrical and spherical coordinates.

9. What is the need to have the general differential equation for heat conduction in three separate coordinate systems? Give their applications.

10. What is meant by 'Initial condition' and 'Boundary Condition'?

11. Explain the B.C.'s of first, second and third kinds. Represent them mathematically.

12. Write down the mathematical formulation of the B.C.'s for heat conduction in a rectangular region $0 \le x \le a, 0 \le y \le b$, for:

(i) Boundary at $x = 0$: heat removed at constant rate of q_0 (W/m^2)

(ii) Boundary at $x = a$: heat dissipation by convection with heat transfer coefficient h_a into the ambient air at constant temperature T_a

(iii) Boundary at $y = 0$: maintained at a constant temperature T_0

(iv) Boundary at $y = b$: heat supplied into the medium at a rate of q_b (W/m^2)

13. Write down the B.C. for the case of a cylindrical wall with inside radius r_1 and outside radius r_2, when the inside surface is heated uniformly at a rate of q (W/m^2) and the outside surface dissipates heat by convection with a heat transfer coefficient h_2 (W/(m^{2}C) into the ambient air at zero deg.C.

14. A spherical shell, inside radius r_1, outside radius r_2, is heated at the inner surface electrically at a rate of q_1 (W/m^2); outside surface dissipates heat by convection with a convection heat transfer coefficient h_2, into ambient at temperature T_a. Write down the B.C.'s.

Problems

1. A wall, 1.5 m thick has the following temperature distribution:

$T(x) = 60 + 18x - 6x^3$ where x is in metres and $T(x)$ is in deg. C. Determine the location of maximum temperature and the heat flow per m^2 area at both the faces. Take $k = 25$ W/(mC). Also, find out the average temperature of the wall.

2. Consider a plane wall 2 cm thick, with uniformly distributed heat sources (q_g, W/m^3) inside its volume; its left and right faces are maintained at temperatures T_1 and T_2, respectively. Steady state temperature distribution in this wall is given by:

$T(x) = 160 - 1000x - 10^5 x^2$. If $q_g = 40$ MW/m^3, determine:

(i) Temperatures T_1 and T_2

(ii) Heat flux at the left face

(iii) Heat flux at the right face

(iv) Heat flux at the centre of the plate

(v) Average temperature of the plate.

3. Temperature distribution in a slab of 1 m thickness is given by:

$T(x) = 900 - 300x - 50x^2$. Heat transfer occurs across an area of 10 m^2 and there is uniform heat generation at a rate of $q_g = 1000$ W/m^3. Assume density $\rho = 1600$ kg/m^3, thermal conductivity $k = 45$ W/(mK) and specific heat $c_P = 4$ kJ/(kgK). Calculate:

(i) Maximum temperature in the slab

(ii) Energy entering the left face (i.e. at $x = 0$)

(iii) Energy leaving the wall at right face (i.e at $x = 1$ m)

(iv) Rate of change of energy storage in the slab, and

(v) Time rate of temperature change at $x = 0.5$ m in the slab.

4. The temperature distribution across a large concrete slab 50 cm thick, heated from one side, as measured by thermocouples approximates to the relation:

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$T(x) = 60 - 50x + 12x^2 + 20x^3 - 15x^4$, where T is in deg. C and x is in metres. Considering an area of 5 m^2, compute:

    (i) heat entering and leaving the slab in unit time

    (ii) heat energy stored in unit time

For concrete, take $k = 1.2$ W/(mK).   [V.T. U., Jan./Feb. 2003]

5. A hollow cylinder of inner radius r_1 and outer radius r_2 has temperature variation along the radius given by:

    $T(r) = 400 - 400.\ln(r/r_1)$. Thermal conductivity of the material, $k = 45$ W/(mC).

    If $r_1 = 5$ cm and $r_2 = 10$ cm, determine the direction and rate of flow of heat at the two surfaces for 1 m length of pipe.

6. A hollow sphere of inner radius r_1 and outer radius r_2 has the temperature along the radius varying as:

    $T(r) = 400 + 400.\ln(r/r_2)$. Assume $k = 45$ W/(mK). If $r_1 = 5$ cm and $r_2 = 10$ cm, determine the direction and rate of flow of heat at the two surfaces. Also, find out the average temperature of the sphere.

7. A 5 cm diameter cylindrical rod ($k = 15$ W/(mC)), with a uniform heat generation rate of q_g (W/m^3) inside it, has a radial temperature distribution given by:

    $T(r) = 315 - 2.1 \times 10^4 \, r^2$ where T is in deg. C, r in metres. Determine:

    (i) Maximum temperature in the rod

    (ii) Volumetric rate of heat generation

    (iii) Average temperature of the cylinder.

8. The steady state radial temperature profile in a 10 cm diameter solid sphere is given by:

    $T(r) = 101.4 - 1390 \, r^2$, where T is in deg. C and r, in metres. Its $k = 10$ W/(mC). The sphere is placed in an ambient of 30°C.

    (a) What is the maximum temperature in the sphere?

    (b) Is there heat generation in the sphere? If yes, at what rate?

    (c) Calculate the convection coefficient at the outer surface.